Density of linear spaces generated by translates of powers of a continuous function

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Hiermit versichere ich, die Arbeit selbstständig verfasst zu haben, und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

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Contents

We will investigate the density of linear spaces generated by translates of powers of a fixed compactly supported, continuous function on a locally compact topological group or more generally certain homogeneous spaces inside different classical function spaces. Such questions have been investigated by Kerman and Weit ([1], [2]) and Ray and Sarkar ([3], [4]). We will summarise most of their results including proofs.

In section 2 we will consider limits of a sequence of normalised powers of the function we want to analyse. Properties of those limits can be used to prove desired density results. For example we will prove density results in the case that a non-negative, non-zero function attains its maximum at a unique point by using the weak-*-convergence of the normalised powers to a Dirac measure. Contrary to the remaining sections, we will consider spaces generated by *two-sided* translates.

Section 3 will provide sharp characterisations of functions generating certain dense subspaces in the case of a locally compact abelian or compact group by considering the Fourier transform of certain characteristic functions associated to the function.

For the special case of a function attaining a maximum at a unique point we will present a result with weaker conditions in section 4: We will consider functions on certain homogeneous spaces and it will suffice to assume a unique maximum which does not have to be an absolute maximum.

In section 5 we will not only consider a linear, (left-)translation invariant space generated by powers of a function, but minimal, left-translation invariant *algebras* and obtain a sharp result about the density of such algebras—based on the Stone-Weierstraß theorem. This result also gives a simple condition for non-density of the linear spaces considered in the previous sections.

The theorems attributed to Kerman, Weit, Ray and Sarkar sometimes differ in some details from the statements in the original articles.

1 Preliminaries

We will consider locally compact spaces, usually denoted by X, or more specifically locally compact topological groups, usually denoted by X, and special cases of homogeneous spaces G/H given by a locally compact group G and a compact subgroup H. A locally compact space is always assumed to be Hausdorff. e denotes the neutral element of a group G. To every locally compact group G we will associate a fixed Haar-measure μ_G (which is unique up to a constant). For a homogeneous space G/H where H is a compact subgroup there exists a left-invariant Radon measure $\mu_{G/H}$, which is also unique up to a constant. In the case of a compact homogeneous space we will assume that $\mu_{G/H}(G/H) = 1$. Every locally compact group G can be regarded as a homogeneous space $G/\{e\}$. On such spaces $X, G/H$ we will consider the following function spaces of real-valued functions:

- $C_c(X)$: Continuous, compactly supported functions with the uniform norm.
- $C_0(X)$: The Banach space of continuous functions vanishing at infinity with the uniform norm, a completion of $C_c(G)$. The dual space is given by finite signed Radon measures ([5], chapter 4, p. 56).
- $C_b(X)$: The Banach space of bounded continuous functions with the uniform norm.
- $C(X)$: The locally convex space of all continuous functions on G with the topology of uniform convergence on compact sets (which is the same as the compact-open topology). The dual space is given by compactly supported signed Radon measures ([5], chapter 4, p. 156).
- $L^p(G/H)$: Banach spaces defined using the Haar measure on G/H. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ the spaces $L^p(G/H) \cong L^q(G/H)'$. $L^\infty(G/H)$ is isomorphic to the dual space of $L^1(G/H)$.

Spaces of complex-valued functions will be denoted as $C(X, \mathbb{C})$ etc., $\mathbb{K}$ is always either $\mathbb{R}$ or $\mathbb{C}$ and $C(X, \mathbb{K})$ etc. denote the corresponding function spaces.

f will also denote a compactly supported, continuous, real-valued function on a given space. For the analysis we will define some sets associated to such a function $f \in C_c(X)$:

- $\mathring{\operatorname{supp}}(f) := f^{-1}(\mathfrak{C}\{0\})$, where $\mathfrak{C}$ denotes complementation.
- $\operatorname{supp}(f) := \overline{\mathring{\operatorname{supp}} f}$.
- $A_\lambda^n := \{x \in G \mid f^n(x) \geq \lambda\} \cap \mathring{\operatorname{supp}}(f)$ for $0 \leq n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. A_λ^n is precompact.
- $A_\lambda := A_\lambda^1$.
- $A_{\lambda_1, \lambda_2} := f^{-1}((\lambda_1, \lambda_2)) \cap \mathring{\operatorname{supp}}(f)$, which is open and precompact.

Given a finite, signed Radon measure $\lambda \in C_0(G)'$, we define:

- λ^* by $\lambda^*(E) = \overline{\lambda(E^{-1})}$, for every measurable set $E \subset G$, where E^{-1} denotes pointwise complementation.
- λ^* by $\tilde{\lambda}(E) := \lambda(E^{-1})$.

For a function f on a group G we define $\check{f}(x) := f(x^{-1})$.

$\mathbb{T}$ will denote the circle group used as a simple example: $\mathbb{T} := \{x \in \mathbb{C} \mid |x| = 1\}$, the operation is given by multiplication of complex numbers.

1.1 Convolutions and translates

In some of the proofs convolutions play an important role. In this section we will consider some relationships between translates of functions and convolutions, which we will later use. In this section G will always denote a locally compact group and H a compact subgroup of G.

Given a function f on a space G/H and $x \in G$: The left-translate $L_x f$ is defined by

$$L_x f(yH) := f(x^{-1}yH).$$

For functions on a group G we can also define a right-translate $R_x f$ by

$$R_x f(y) := f(yx).$$

For some arguments convolutions will play an important role. Given two Radon-measures $\mu, \nu \in C_0(G)'$, the convolution $\mu * \nu$ is defined by

$$(\mu * \nu)(f) := \int \int f(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

Using this definition we can deduce various expressions for convolutions of functions (by regarding functions f as measures $f\,\mathrm{d}\mu_G$) and measures:

$$\begin{aligned}(\mu * f)(y) &= \int_G f(x^{-1}y)\,\mathrm{d}\mu(x) = \mu(L_y \check{f}) &&= \textstyle\int_G (L_x f)(y)\,\mathrm{d}\mu(x)\\ &= \int_G (R_y f)(x^{-1})\,\mathrm{d}\mu(x) &&= \tilde{\mu}(R_y f)\end{aligned}$$

$$(f * \mu)(x) = \int f(xy^{-1})\Delta_G(y^{-1})\,\mathrm{d}\mu(y) = \mu(\Delta_G^{-1} R_{x^{-1}} \check{f}) = \tilde{\mu}((L_{x^{-1}} f)\cdot \Delta_G)$$

$$\begin{aligned}(f * g)(x) &:= \int_G f(xy)g(y^{-1})\,\mathrm{d}y = \int_G (L_{x^{-1}} f)(y) g(y^{-1})\,\mathrm{d}y\\ = \int_G f(y) L_y g(x)\,\mathrm{d}y &= \int_G f(y) L_x \check{g}(y)\,\mathrm{d}y = \int_G \Delta_G^{-1}(y)\check{f}(y) R_x g(y)\,\mathrm{d}y\end{aligned}$$

Theorem 1. *Consider a closed subspace $\mathcal{I} \subset L^1(G)$. $\mathcal{I}$ is a left-ideal in the group algebra $L^1(G)$ if and only if it is left-translation invariant ([6], p. 54).*

This theorem includes the statement that convolutions can be approximated by linear combinations of translates.

Lemma 2. *Consider $f, g \in L^1(G)$. $f * g$ can be approximated in L^1 by linear combinations of left-translates of g.*

Proof. Let $\mathcal{I} \subset L^1(G)$ denote the closed subspace generated by all left-translates of g. $\mathcal{I}$ is left-translation invariant, thus $f * \mathcal{I} = \mathcal{I}$ and $f * g \in \mathcal{I}$. □

We can prove a similar statement for convolutions of continuous functions:

Lemma 3. *If $g \in C_0(G), f \in C_c(G)$, then $g * f \in C_0(G)$ and $g * f$ can be uniformly approximated by linear combinations of left-translates of f.*

Proof. Let $\varepsilon > 0$. For all $x \notin g^{-1}(\mathfrak{C}(-\varepsilon, \varepsilon)) \cdot \operatorname{supp}(f) =: D$ we have $|(g * f)(x)| \leq \varepsilon C$, where $C > 0$ only depends on f, this gives us $g * f \in C_0(G)$. Now $E := D \cdot \operatorname{supp}(f)^{-1}$ is a compact set and $a : E \to C_0(G), y \mapsto L_y f$ is a continuous function. Then the vector-valued integral $\int_E g(y)a(y)\,\mathrm{d}y$ exists and can be approximated by linear combinations of left-translates of f ([6], p. 260, theorem A3.3). Choose such an ε-approximation $h \in C_c(G)$. For $x \in D$ we get $\int_E g(y)a(y)(x)\,\mathrm{d}y = (g * f)(x)$, thus $\|(g * f - h) \restriction D\|_\infty < \varepsilon$. For $x \notin D$ we get $\left|\int_E g(y)a(y)(x)\,\mathrm{d}y\right| \leq \varepsilon C$, if C had been choosen appopriately. This implies that $\|h - g * f\|_\infty \leq \varepsilon(2C + 1)$. □

1.2 The Fourier transform

We will subsume some facts regarding the Fourier transform.

Lemma 4. *Every unitary (strongly/weakly continuous) representation π of G on a Hilbert space $\mathcal{H}_\pi$ can be regarded as a *-homomorphism $C_0(G)' \to L(\mathcal{H}_\pi)$ using the definition*

$$\langle \pi(\mu)u, v\rangle := \int \langle \pi(x)u|v\rangle\,\mathrm{d}\mu(x)$$

for $\mu \in C_0(G)'$ and $u, v \in \mathcal{H}_\pi$.

Proof. Linearity is obvious. Consider two measures $\mu, \nu \in C_0(G)'$ and $u, v \in \mathcal{H}_\pi$, we check:

$$\pi(\mu * \nu) = \int\int \pi(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = \int\int \pi(x)\,\mathrm{d}\mu(x)\pi(y)\,\mathrm{d}\nu(y) = \pi(x)\pi(y)$$

$$\langle \pi(\mu^*)u, v\rangle = \int \langle \pi(x)u, v\rangle\,\mathrm{d}\mu^*(x) = \overline{\int \overline{\langle \pi(x)u, v\rangle}\,\mathrm{d}\mu(x^{-1})} = \overline{\int \langle v, \pi(x)u\rangle\,\mathrm{d}\mu(x^{-1})}$$

$$= \overline{\int \langle \pi(x^{-1})v, u\rangle\,\mathrm{d}\mu(x^{-1})} = \overline{\langle \pi(\mu)v, u\rangle} = \langle \pi(\mu)^* u, v\rangle$$

$\pi(\mu)$ is bounded by $\|\pi(\mu)\| \leq \|\mu\|$ (with respect to the operator norm), since

$$\max_{u \in \mathcal{H}_\pi, \|u\|=1} \|\pi(\mu)u\| = \max_{u,v \in \mathcal{H}_\pi, \|u\|=1=\|v\|} \langle \pi(\mu)u, v\rangle = \int \langle \pi(x)u, v\rangle\,\mathrm{d}\mu(x) \leq \|\mu\|$$

using the unitarity of the representation π. □

Given a maximal set $\hat{G}$ of pairwise inequivalent, irreducible unitary representations of a locally compact group G, the *Fourier transform* of a measure $\mu \in C_0(G)'$ is given by the family $(\pi(\mu))_{\pi \in \hat{G}}$.

Theorem 5. *The Fourier transform is injective on the space $C_0(G)'$ of finite measures ([7], p. 356).*

1.3 The subspaces at hand

Now let us introduce the spaces whose density we want to characterise: Given a function $f \in C_c(G/H)$ and a set $A \subset \mathbb{N}$ we define

$$V_A(f) := \operatorname{span}\{L_x f^n \mid x \in G, n \in A\},$$

where we define $f^0(x) := 1$. $V_0(f) := V_{\mathbb{N}}(f)$ and $V(f) := V_{\mathbb{N}\setminus\{0\}}(f)$. Let $A(f)$ denote the minimal left-translation invariant sub-algebra of $C(G)$ containing f. If $0 \notin A$, then $V_A(f) \subset C_c(G/H)$.

In the case of a function $f \in C_c(G)$ on a group G we can also consider spaces of two-sided translates: Let $W_A(f)$ denote the minimal left- and right-translation invariant subspace of $C(G)$ containing $\{f^n \mid n \in A\}$, $W(f) := W_{\mathbb{N}\setminus\{0\}}(f)$. We notice that $W_A(f) \subset C_c(G)$ if $0 \notin A$.

2 Weak-*-convergence of the normalised powers

Definition 1. For a function $f \in C_c(G, \mathbb{C})$ on a locally compact group G we define a sequence $(f_{(n)})_{n>0}$ by

$$f_{(n)} := \frac{f^n}{\int |f^n|}.$$

Lemma 6. *For $f \in C_c(G, \mathbb{K})$ the sequence $(f_{(n)}\, \mathrm{d}\mu_G)$ has a cluster point $\nu \in C(G, \mathbb{K})'$ with respect to the weak-*-topology. For every such cluster point* $\operatorname{supp}(\nu) \subset \operatorname{supp}(f)$.

Proof. The norm of every measure $f_{(n)}\, \mathrm{d}\mu_G$ is obviously less than 1. By the Banach-Alaoglu theorem the closed unit ball in the space $C_0(G, \mathbb{K})'$ is compact with respect to the weak-*-topology, thus the sequence has a cluster point ν in $C_0(G, \mathbb{K})'$ with the weak-*-topology. Now we claim that $\operatorname{supp}(\nu) \subset \operatorname{supp}(f)$. Consider any $g \in C_c(G, \mathbb{K})$ such that $\operatorname{supp}(g) \cap \operatorname{supp}(f) = \varnothing$. Then $\int g f_{(n)}\, \mathrm{d}\mu_G = 0$ for all $n > 0$ implying that $\nu(g) = 0$. Since $\operatorname{supp}(f)$ is compact, ν is compactly supported, thus $\nu \in C(G, \mathbb{K})'$. We still have to prove that ν is even a cluster point of the sequence in the space $C(G, \mathbb{K})'$ with the weak-*-topology: Let $g_1, \ldots, g_m \in C(G, \mathbb{K})$ and $\varepsilon > 0$. By Urysohn's lemma there are functions $h_1, \ldots, h_m \in C_c(G)$ such that $g_i \restriction \operatorname{supp}(f) = h_i \restriction \operatorname{supp}(f)$. Since ν is a cluster point in $C_0(G)'$ with the weak-*-topology, there exist infinitely many $n > 0$ such that for all $1 \leq i \leq m$ $\left|(\nu - f_{(n)}\, \mathrm{d}\mu_G)(h_i)\right| < \varepsilon$. But $\nu(h_i) = \nu(g_i)$ and $\int h_i f_{(n)}\, \mathrm{d}\mu_G = \int g_i f_{(n)}\, \mathrm{d}\mu_G$. Thus ν is even a cluster point with respect to the weak-*-topology of $C(G, \mathbb{K})'$. $\square$

Lemma 7. *In the situation of the previous lemma assume that f is real-valued,* $\max f > 0$ *and* $-\min f < \max f$*. For every such ν as in the previous lemma we get that* $\operatorname{supp}(\nu) \subset A_{\max f}$*.*

Proof. Consider any function $g \in C_c(G)$ such that $\operatorname{supp}(g) \cap A_{\max f} = \emptyset$. Define

$$r := \max_{x \in \operatorname{supp}(g)} |f(x)| < \max f$$

and $s := \frac{r + \max f}{2} > r$. For $n > 0$ we get the approximation

$$\left| \int f_{(n)} g \, \mathrm{d}\mu_G \right| \leq |\operatorname{supp}(f)| \cdot \left(\frac{r^n}{\int |f|^n} \right) \leq |\operatorname{supp}(f)| \cdot \left(\frac{r}{|A_s| \cdot s^n} \right) = \frac{|\operatorname{supp}(f)|}{|A_s|} \cdot \left(\frac{r}{s} \right)^n.$$

This expression gets arbitrarily small for sufficiently large n, thus any weak-*-cluster point ν must fulfill $\nu(g) = 0$. $\square$

Lemma 8. *Given any $\nu \in C(G, \mathbb{K})'$. If the map $C_b(G, \mathbb{K}) \to C_b(G, \mathbb{K}), h \mapsto h * \nu$ is injective, then the subspace W spanned by all right-translates of $C_0(G, \mathbb{K}) * \nu$ is dense in $C_0(G, \mathbb{K})$. If additionally $\nu \in L^1(G, \mathbb{K})$, then for all $1 \leq p < \infty$ the subspace W_p spanned by all right-translates of $L^p(G, \mathbb{K}) * \nu$ is dense in $L^p(G, \mathbb{K})$.*

Proof. Assume the contrary. Choose $g \in C_c(G, \mathbb{K}) \setminus \overline{W}$ and—using Hahn-Banach—a measure $\mu \in C_0(G, \mathbb{K})'$ such that μ vanishes on W and $\mu(g) \neq 0$. Then for every $x \in G$ we get $0 = \mu(R_x(g * \nu)) = (\tilde{\mu} * (g * \nu))(x)$, thus $(\tilde{\mu} * g) * \nu = 0$. But $\tilde{\mu} * g \in C_b(G, \mathbb{K})$, thus the map $h \mapsto h * \nu$ is not injective.

Now consider $b := \nu \in L^1(G, \mathbb{K})$. Assume that W_p is not dense in $L^p(G, \mathbb{K})$. Choose $g \in C_c(G, \mathbb{K}) \setminus \overline{W_p}$ and—using Hahn-Banach—a function $a \in L^q(G, \mathbb{K})$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that $\int ag \neq 0$ and $\int al = 0$ for all $l \in W_p$. Now for every $x \in G$ we get $0 = \int \check{a} \Delta_G^{-1} R_x(g * b) = (a * (g * b))(x)$ (notice that the integrals are well-defined since $g * b \in C_c(G, \mathbb{K})$), thus $(a * g) * b = 0$. $a * g \in C_b(G, \mathbb{K})$ because $g \in C_c(G, \mathbb{K})$. Again, we get that $h \mapsto h * b$ is not injective. $\square$

Lemma 9. *Let G be a compact group. If for every irreducible unitary representation π of G $\pi(\nu)$ is surjective, then the map $C(G, \mathbb{K}) \to C(G, \mathbb{K}), g \mapsto g * \nu$ is injective.*

Proof. Consider any $0 \neq g \in C(G, \mathbb{K})$. Since the Fourier transform is injective (theorem 5), there exists an irreducible unitary representation π such that $\pi(g) \neq 0$. Then $\pi(g * \nu) \neq 0$, since $\pi(\nu)$ is surjective. We conclude that $g * \nu \neq 0$. $\square$

Lemma 10. *Let $\nu \in C(G, \mathbb{K})'$ be any weak-*-cluster point of the sequence $(f_{(n)} \, \mathrm{d}\mu_G)$. If the subspace spanned by all right-translates of $C_0(G, \mathbb{K}) * \nu$ is dense in $C_0(G, \mathbb{K})$, then $W_A(f)$ is dense in $C_0(G, \mathbb{K})$ for any cofinite $A \subset \mathbb{N}$. If ν is even a limit of the sequence, then every $W_B(f)$ where $B \subset \mathbb{N}$ is infinite is dense in $C_0(G, \mathbb{K})$.*

Proof. Let $g \in C_0(G, \mathbb{K})$. For a given $0 < n \in \mathbb{N}$ and $x \in G$ we define

$$\psi_n(x) := \left|\left(g * f_{(n)} - g * \nu\right)(x)\right| = \left|\int \Delta_G^{-1} R_{x^{-1}} \check{g}(f_{(n)}\, \mathrm{d}\mu_G - \mathrm{d}\nu)\right|.$$

Each ψ_n is continuous. Let $\varepsilon > 0$. Since g vanishes at infinity, there exists a compact set $X \subset G$ such that for all $n > 0$ and all $x \notin X$ $\psi_n(x) < \varepsilon$. For $x \in X$ let n_x denote the minimal positive integer such that $\psi_{n_x}(x) < \varepsilon$. $\left\{\psi_{n_x}^{-1}([0, \varepsilon)) \mid x \in X\right\}$ is an open cover of X. There exists a finite subcover labelled by $x_1, \ldots x_m \in X$. Because ν is a cluster point and every $\Delta_G^{-1} R_{x_i^{-1}} \check{g}$ is a continuous function, there exist arbitrary large $n > 0$ (thus we can choose $n \in A$) such that $\psi_n(x_i) < \varepsilon$ for $1 \le i \le m$. This implies $\|g * f_{(n)} - g * \nu\|_\infty \le \varepsilon$. In the special case that $(f_{(n)}\, \mathrm{d}\mu_G)$ converges to ν, every sufficiently large n satisfies this condition, thus we can choose $n \in B$.

By lemma 3 there is a linear combination of left-translates of $f_{(n)}$, $h \in C_c(G, \mathbb{K})$, such that $\|h - g * f_{(n)}\| < \varepsilon$. We combine our approximations and get $\|h - g * \nu\|_\infty < 2\varepsilon$. We have proved that $V_A(f)$ (or $V_B(f)$, respectively) is dense in $C_0(G, \mathbb{K}) * \nu$ and conclude that $W_A(f)$ (or $W_B(f)$, respectively) is dense in $C_0(G, \mathbb{K})$. □

Lemma 11. *If $b \in L^1(G, \mathbb{K})$ is an L^1-cluster point of the sequence $(f_{(n)})$ and the subspace spanned by all right-translates of $L^1(G, \mathbb{K}) * b$ is dense in $L^1(G, \mathbb{K})$, then $W(f)$ is dense in $L^1(G, \mathbb{K})$.*

Proof. This is an immediate consequence of lemma 2. □

To prove density in concrete cases it is useful to have some general lemmata providing limits of the sequence $(f_{(n)})$. In the rest of the section we will assume that f is real-valued, $\max f > 0$ and $-\min f < \max f$.

We will need the following measure-theoretic lemma:

Lemma 12. *Consider a sequence $(X_n)_{n \in \mathbb{N}}$ of subsets of a measure space (X, μ) such that every $\mu(X_n)$ is finite. If $X_n \supset X_{n+1}$ holds for all $n \in \mathbb{N}$, then $l := \lim_{n\to\infty} \mu(X_n) = \inf_{n \in \mathbb{N}} \mu(X_n) = \mu(X) := \mu(\bigcap_{n \in \mathbb{N}} X_n)$.*

Proof. Define the measurable sets $Y_n := X_n \setminus X_{n+1}$. This family is pairwise disjoint and $X_0 \setminus X = \bigcup_{n \in \mathbb{N}} Y_n$. By σ-additivity we get

$$\begin{aligned} \mu(X) &= \mu(X_0 \setminus \bigcup_{n \in \mathbb{N}} Y_n) = \mu(X_0) - \sum_{n \in \mathbb{N}} \mu(Y_n) \\ &= \mu(X_0) - \lim_{n \to \infty} \mu(\bigcup_{i<n} Y_i) = \mu(X_0) - \lim_{n\to\infty} (X_0 \setminus X_n) = \lim_{n\to\infty} \mu(X_n). \end{aligned}$$

□

Lemma 13. *If $|A_{\max f}| > 0$, then the sequence $(f_{(n)})$ converges to $\chi_{A_{\max f}} \cdot |A_{\max f}|^{-1}$ in L^1.*

Proof. Without loss of generality assume that $\max f = 1$ (the functions $f_{(n)}$ are invariant with respect to scaling of f). Let $\varepsilon > 0$. By lemma 12 there exists $\delta > 0$ such that $|A_{1-\delta} \setminus A_1| < \frac{\varepsilon \cdot |\operatorname{supp} f|}{2}$. For sufficiently large $n > 0$ we get $\max_{x \notin A_{1-\delta}} f_{(n)}(x) \le \max_{x \notin A_{1-\delta}} \frac{f^n(x)}{|A_1|} < \frac{\varepsilon}{2}$ since $|A_1| \le \|f^n\|_{L_1}$. Using $\max f_{(n)} \le \frac{1}{\operatorname{supp} f}$ we get

$$\int_{G \setminus A_1} |f_{(n)}| = \int_{G \setminus A_{1-\delta}} |f_{(n)}| + \int_{A_{1-\delta} \setminus A_1} |f_{(n)}| < \max f \cdot \frac{\varepsilon \cdot |\operatorname{supp} f|}{2} + \frac{\varepsilon}{2} \le \varepsilon.$$

Now we can conclude that $\max f_{(n)} = \frac{1}{|A_1| + \int_{G \setminus A_1} |f^n|}$ converges to $\frac{1}{|A_1|}$ for $n \to \infty$ (since $\int_{G \setminus A_1} |f^n| \le \int_{G \setminus A_1} |f_{(n)}| \cdot |\operatorname{supp} f|$). Thus for sufficiently large $n > 0$ the inequalities $\int_{G \setminus A_1} |f_{(n)}| < \varepsilon$ and $\int_{A_1} \left(f_{(n)} - \chi_{A_1} \cdot |A_{\max f}|^{-1} \right) < \varepsilon$ hold and we get the desired result. □

Lemma 14. *If f attains its maximum at a unique point x_0, then $(f_{(n)} \, \mathrm{d}\mu_G)$ converges to the Dirac measure δ_{x_0} in the weak-*-topology.*

Proof. Consider any $g \in C(G)$. $\int (f_{(n)} - \delta_{x_0}) g = \left(\int f_{(n)} g \right) - g(x_0) = \int f_{(n)} \cdot (g - g(x_0))$, thus it suffices to prove the convergence of $\int f_{(n)} g$ to 0 for $g(x_0) = 0$. Thus assume $g(x_0) = 0$ and consider any $\varepsilon > 0$. Define $E := \{x \in G \mid |g(x)| < \varepsilon\}$. $\int_E f_{(n)} g \le \varepsilon$ and for sufficiently large n also $\int_{\complement E} f_{(n)} g < \varepsilon$, since E is a neighbourhood of x_0. □

Corollary 15. *If f attains its maximum at a unique point, then $W(f)$ is dense in $C_0(G)$.*

Proof. The map $C_b(G) \to C_b(G), h \mapsto h * \delta_{x_0}$ is obviously surjective, since

$$(g * \delta_{x_0})(x) = \int f(xy^{-1}) \Delta_G(y^{-1}) \, \mathrm{d}\delta_{x_0}(y) = f(x x_0^{-1}) \Delta_G(x_0^{-1}) = \Delta_G(x_0^{-1})(R_{x_0} f)(x).$$

Using lemma 10 we conclude that $W_B(f)$ is dense in $C_0(G)$ for every infinite set $B \subset \mathbb{N}$. □

Example 1. Assume that $\max f = 1$. If A_1 is finite, then for every point $x \in A_1$ we can choose a function $a_x \in C(G)$ such that $\operatorname{supp} a_x \subset \operatorname{supp} f$, there exists a neighbourhood U_x of x satisfying $a_x \restriction U_x = 1$, $\|a_x\|_\infty = 1$ and the sets $\operatorname{supp} a_x$ are pairwisely disjoint for $x \in A_1$ (use Urysohn's lemma). Consider $f' := \sum_{x \in A_1} a_x \cdot f$. Since f and f' coincide on a neighbourhood of A_1, the limits of $(f_{(n)})$ and $(f'_{(n)})$ coincide. For any $x \in A_1$ consider a sequence given by

$$s_n^x := \frac{\int |a_x \cdot f|^n}{\int |f'|^n}.$$

According to lemma 14 $\left((a_x \cdot f)_{(n)} \, \mathrm{d}\mu_G \right)$ converges to δ_x in the weak-*-topology. If $(s_n^x)_{n>0}$ converges to a value S^x for each $x \in A_1$, then $(f'_{(n)})$ (and thus $(f_{(n)})$) converges to

$$\sum_{x \in A_1} S^x \delta_x$$

in the weak-*-topology, since

$$f'_{(n)} = \frac{f'^n}{\int |f'|^n} = \frac{\sum_{x \in A_1} (a_x \cdot f)^n}{\int |f'|^n} = \sum_{x \in A_1} s_n^x (a_x \cdot f)_{(n)}$$

Let us consider the circle group $\mathbb{T}$ and a function $f \in C(\mathbb{T})$ given by $f = c_0 + c_1$ where the functions c_i are given by

$$c_i(x) := \begin{cases} \left(1 - \frac{|\arg(x \cdot x_i^{-1})|}{2\pi b_i}\right)^{p_i} & \text{for } |\arg(x \cdot x_i^{-1})| < 2\pi b_i \\ 0 & \text{otherwise} \end{cases}$$

such that the supports of c_0 and c_1 are disjoint. Then (using the normalised Haar measure and choosing $a_{x_i} \cdot f = c_i$)

$$\int |a_{x_i} \cdot f|^n = \int c_i^n = \frac{2b_i}{np_i + 1}$$

implying

$$S^{x_i} = \frac{b_i p_{1-i}}{b_{1-i} p_i + b_i p_{1-i}}.$$

Using previous lemmata we conclude that $W(f) = V(f)$ is dense in $C(\mathbb{T})$ if $x_0^z S^{x_0} + x_1^z S^{x_1} \neq 0$ for all $z \in \mathbb{Z}$: Every irreducible representation of $\mathbb{T}$ is given by $\pi_z : x \mapsto x^z$ for a $z \in \mathbb{Z}$. The limit of the sequence $(f_{(n)}\, \mathrm{d}\mu_G)$ is given by $\gamma := S^{x_0}\delta_{x_0} + S^{x_1}\delta_{x_1}$. If $\pi_z(\gamma) = x_0^z S^{x_0} + x_1^z S^{x_1}$ is non-zero for every $z \in \mathbb{Z}$, then the map $C(\mathbb{T}) \to C(\mathbb{T}), g \mapsto g * \gamma$ is injective by lemma 9. The lemmata 8 und 10 imply that $W_A(f)$ is dense in $C(\mathbb{T})$ for every infinite $A \subset \mathbb{N}$.

Example 2. Let f be a function on the circle group such that $A_{\max f}$ is an interval starting at e. $(f_{(n)})$ converges (in L^1) to a function ψ given by

$$\psi(x) = \begin{cases} \frac{1}{b} & \text{for } \arg(x) < 2\pi b \\ 0 & \text{otherwise} \end{cases}$$

for some $b > 0$. If b is irrational, then $\pi(\psi) \neq 0$ for every irreducible representation π of $\mathbb{T}$. By lemma 9 the map $C(\mathbb{T}) \to C(\mathbb{T}), g \mapsto g * \psi$ is injective. Again we get density of $W_A(f)$ in $C(\mathbb{T})$ using the lemmata 8 and 10.

3 Characterisation for locally compact abelian groups and compact groups using the Fourier transform of characteristic functions

Lemma 16. *Let G be a compact group. Let $\lambda \in C(G)'$. If V is a left-translation invariant subspace of* $\ker \lambda$, *then $f * \tilde{\lambda} = 0$ for $f \in V$.*

Proof. We use unimodularity:

$$(f * \tilde{\lambda})(x) = \lambda((L_{x^{-1}}f) \cdot \Delta_G) = \lambda(L_{x^{-1}}f) = 0$$

□

For locally compact abelian groups there is a well-known theorem by Wiener characterising dense, translation invariant subspaces:

Theorem 17. *Let G be a locally compact abelian group. A translation invariant subspace $V \subset L^1(G)$ is dense in $L^1(G)$ if and only if there is no $\pi \in \hat{G}$ such that $\pi(f) = 0$ for all $f \in V$ ([6], theorem 4.63) Since every $\pi \in \hat{G}$ is linear on $L^1(G)$ and $\pi(L_x f) = \pi(x)\pi(f)$ it suffices to consider a subset of V whose translates span the space V.*

For compact groups a similar theorem can be stated in the space of continuous functions, which is a simple generalisation of proposition 1 in [1].

Theorem 18. *Let G be a compact group. A left-translation invariant subspace $V \subset C(G)$ is dense in $C(G)$ if and only if there is no $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$ such that for all $f \in V$ $\pi(f)v = 0$. Again, it is sufficient to consider a subset of V whose translates span V.*

Proof. Assume there exists a $\pi \in \hat{G}$, $v \in \mathcal{H}_\pi$ such that $\pi(f)v = 0$ for all $f \in V$. Now consider the measure λ given by $\int g(x)\,\mathrm{d}\lambda(x) = \int g(x)\langle\pi(x)v, v\rangle\,\mathrm{d}\mu_G(x) \in C(G)'$. λ is non-zero because $x \mapsto \langle\pi(x)v, v\rangle$ is continuous and non-zero at $x = e$: Choose a positive, non-zero continuous function g supported on a neighbourhood of e where $x \mapsto \langle\pi(x)v, v\rangle$ is positive. Then $\lambda(g) > 0$. Now consider $f \in V$:

$$\lambda(f) = \int f(x)\langle\pi(x)v, v\rangle\,\mathrm{d}x = \langle\pi(f)v, v\rangle = 0$$

Thus $\overline{V} \subset \ker\lambda \subsetneq C(G)$.

Now assume $\overline{V} \subsetneq C(G)$. By the Hahn-Banach theorem there exists a non-zero measure $\lambda \in C(G)'$ annihilating all $f \in V$. By theorem 5 there exist $\pi \in \hat{G}, u \in \mathcal{H}_\pi$ such that $\pi(\tilde{\lambda})u \neq 0$. But for all $f \in V$ we conclude using lemma 16 that $0 = \pi(f * \tilde{\lambda}) = \pi(f)\pi(\tilde{\lambda})$. Thus $\pi(f)$ annihilates $\pi(\tilde{\lambda})u$, which does not depend on f. □

Lemma 19. *Let π be a unitary representation of a locally compact group G and $v \in \mathcal{H}_\pi$. Given a compactly supported function $f \in C_c(G)$. If $\pi(\chi_{A_\lambda})v = 0$ for all $\lambda \in \mathbb{R}$, then also $\pi(\chi_{f^{-1}\{\lambda\}})v = 0 = \pi(\chi_{A_{\lambda_1,\lambda_2}})v$ for all $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$ where $\lambda \neq 0$.*

Proof. The continuity of π on $L^1(G)$ implies

$$\pi(\chi_{f^{-1}\{\lambda\}})v = \pi\left(\lim_{\lambda' \nearrow \lambda_1}\left(\chi_{A_\lambda} - \chi_{A_{\lambda'}}\right)\right)v = \lim_{\lambda' \nearrow \lambda}\left(\pi(\chi_{A_\lambda})v - \pi(\chi_{A_{\lambda'}})v\right) = 0.$$

Now consider any $0 \neq \lambda_1 < \lambda_2 \in \mathbb{R}$. $A_{\lambda_1,\lambda_2} = A_{\lambda_1} \setminus \left(A_{\lambda_2} \cup f^{-1}\{\lambda_1\}\right)$ implies

$$\chi_{A_{\lambda_1,\lambda_2}} = \chi_{A_{\lambda_1}} - \chi_{A_{\lambda_2}} - \chi_{f^{-1}\{\lambda_1\}},$$

by linearity of π we get $\pi(\chi_{A_{\lambda_1,\lambda_2}})v = 0$. All considered functions are integrable because f is compactly supported. Similarly, for $\lambda_1 = 0$ we get $A_{\lambda_1,\lambda_2} = A_{\lambda_1} \setminus A_{\lambda_2}$. □

The following theorems will rely on the Weierstraß approximation theorem, or more precisely on the Müntz-Szasz theorem (to give a stronger result) as stated below (see [8], p. 313):

Theorem 20. *Given a subset* $A \subset \mathbb{N}$. *The set of polynomial functions* $\{[0,1] \to \mathbb{R}, x \mapsto x^n \mid n \in A\}$ *spans a dense subspace of* $C([0,1])$ *if and only if* $0 \in A$ *and*

$$\sum_{n \in A} \frac{1}{n} = \infty.$$

Kerman and Weit have proved a result similar to the following one in the special case of the circle group ([1], theorem 1):

Theorem 21. *Let* G *be a compact group,* $f \in C(G)$. *The space* $V(f)$ *is dense in* $C(G)$ *if and only if there is no* $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$ *such that* $\pi(\chi_{A_\lambda})v$ *vanishes for all* $\lambda \in \mathbb{R}$. *If* f *is positive, then even density of* $V_0(f)$ *in* $C(G)$ *implies that there is no such* π, v *If there is no such* π, v *and* $0 \in A \subset \mathbb{N}$ *such that* $\sum_{n \in A} \frac{1}{n} = \infty$, *then* $V_A(f)$ *is dense in* $C(G)$.

As noticed by Weit ([2], theorem 2) we can state a similar theorem for locally compact Abelian groups G regarding density of $V(f)$ in $L^1(G)$ using Wiener's theorem:

Theorem 22. *Given a locally compact abelian group* G, *consider any compactly supported function* $f \in C_c(G)$. $V(f)$ *is dense in* $L^1(G)$ *if and only if there is no* $\pi \in \hat{G}$ *such that for all* $\lambda \in \mathbb{R}$ $\pi(\chi_{A_\lambda}) = 0$.

We will prove these two theorems simultaneously:

Proof. In the compact case we assume without loss of generality that f is positive and $\|f\| = 1$ (in the maximum norm): Otherwise consider $(f + (\|f\|+1) \cdot f^0) / \|f + (\|f\|+1) \cdot f^0\|$. Then it suffices to consider $\lambda \in [0,1]$ ($\chi_{A_\lambda} = \chi_{A_0} = 1$ for $\lambda < 0$ and $\chi_{A_\lambda} = 0$ for $\lambda > 1$).

Suppose that there are $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$ such that $\pi(\chi_{A_\lambda})v = 0$ for all $\lambda \in \mathbb{R}$. Notice that for all $0 < n \in \mathbb{N}$

$$A^n_\lambda = \begin{cases} \{x \in G \mid f(x) \geq \sqrt[n]{\lambda} \wedge f(x) \neq 0\} = A_{\sqrt[n]{\lambda}} & \text{for } n \text{ odd} \\ \mathring{\operatorname{supp}}(f) & \text{for } \lambda < 0 \text{ and } n \text{ even} \\ \{x \in G \mid |f(x)| \geq \sqrt[n]{\lambda} \wedge f(x) \neq 0\} = \mathring{\operatorname{supp}}(f) \setminus A_{-\sqrt[n]{\lambda}, \sqrt[n]{\lambda}} & \text{otherwise} \end{cases},$$

where $\mathring{\operatorname{supp}}(f)$ can be expressed as $A_{\min f}$. In the case $n = 0$ where G s compact and f positive notice that $A_{0,\lambda} = G = A_0$ for $\lambda \leq 1$ and $A_{0,\lambda} = \emptyset = A_{\max f + 1}$ otherwise. Thus using lemma 19 and linearity of π we get $\pi(\chi_{A^n_\lambda})v = 0$ for all $0 < n \in \mathbb{N}, \lambda \in \mathbb{R}$ (and even for $n = 0$ in the compact case). For every $0 < n \in \mathbb{N}$ there exists a net $(g_F)_{0,1 \in F \subset [0,1], |F| \in \mathbb{N}}$ indexed by all finite subsets of the unit interval containing 0 and 1 given by

$$g_{\{0=\lambda_0<\ldots<\lambda_k=1\}} := \sum_{0<i\leq k} (\lambda_i - \lambda_{i-1})\, \chi_{A^n_{\lambda_i}}$$

which converges uniformly to f^n since

$$\|f^n - g\|_\infty \leq \max_{0<i\leq k} (\lambda_i - \lambda_{i-1}).$$

Thus (g_F) also converges to f^n in L^1 (because the supports of all considered functions are contained in the compact set $\operatorname{supp}(f)$) and we get $\pi(f^n)v = 0$ by continuity of π on L^1. Thus, in the compact case, $V(f)$ (and not even $V_0(f)$ if f is positive) is not dense in $C(G)$ by theorem 18. In the abelian case, $V(f)$ is not dense in $L^1(G)$ by Wiener's theorem (17).

For the opposite direction consider any $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$. We assume that there is a $\lambda \in [\min_{x\in G} f(x), \max_{x\in G} f(x)]$ such that $\pi(\chi_{A_\lambda})v \neq 0$. Without loss of generality we can assume $\|v\| = 1$ (otherwise consider $v/\|v\|$). Choose $u \in \mathcal{H}_\pi$ such that $\|u\| = 1$ and $0 < \langle \pi(\chi_{A_\lambda})v, u\rangle \in \mathbb{R}$. We define q as

$$0 < q \coloneqq \langle \pi(\chi_{A_\lambda})v, u\rangle = \int_{A_\lambda} \langle \pi(x)v, u\rangle \,\mathrm{d}x = \int_{A_\lambda} \Re\left(\langle \pi(x)v, u\rangle\right) \mathrm{d}x$$

The integrals exist by precompactness of A_λ and unitarity of $\pi(x)$. There exists $\varepsilon > 0$ such that $|A_{\lambda-\varepsilon,\lambda}| < \frac{q}{2}$ (use lemma 12: $\bigcap_{1\leq n\in\mathbb{N}} A_{\lambda-\frac{1}{n}} = A_\lambda$ and all sets A_μ are precompact because f is compactly supported). Choose a positive function $T \in C([\min_{x\in G} f(x) - \varepsilon, \max_{x\in G} f(x)])$ such that $\operatorname{supp}(T) \subset (\lambda - \varepsilon, \max_{x\in G} f(x)]$, $\|T\| = 1$ and $T(x) = 1$ for $x \geq \lambda$ (it can be choosen piecewisely linear) and define $E \coloneqq T \circ f$. Now we get:

$$\langle \pi(E)v, u\rangle = \int_{A_{\lambda-\varepsilon,\lambda}} E(x)\langle \pi(x)v, u\rangle \,\mathrm{d}x + \int_{A_\lambda} E(x)\langle \pi(x)v, u\rangle \,\mathrm{d}x$$

We can estimate the the first summand by

$$\left|\int_{A_{\lambda-\varepsilon,\lambda}} E(x)\langle \pi(x)v, u\rangle \,\mathrm{d}x\right| \leq \|E\| \cdot \left|A_{\lambda-\varepsilon,\lambda}\right| = \left|A_{\lambda-\varepsilon,\lambda}\right| < \frac{q}{2}.$$

The second summand evaluates to

$$\int_{A_\lambda} E(x)\langle \pi(x)v, u\rangle \,\mathrm{d}x = \int_{A_\lambda} \langle \pi(x)v, u\rangle \,\mathrm{d}x = \langle \pi(\chi_{A_\lambda})v, u\rangle = q.$$

Thus $\langle \pi(E)v, u\rangle > \frac{q}{2}$, especially $\pi(E)v \neq 0$. Assume that $\pi(f^n)v$ vanishes for all $0 < n \in \mathbb{N}$. Then $\pi(P \circ f)v = 0$ for every real polynomial function P defined on $[\min_{x\in G} f(x) - \varepsilon, \max_{x\in G} f(x)]$ with the restriction that the constant term of P is zero in the non-compact case. We will now prove that there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of such polynomials such that $(P_n \circ f)_{n\in\mathbb{N}}$ converges to $T \circ f = H$ in L^1, then by continuity of π on L^1 we get $\pi(H)v = 0$, a contradiction:

Extend the domain of T such that $0 \in \operatorname{dom}(T)^\circ$. By the Weierstraß theorem the polynomial functions where the constant term is zero span a dense subspace of the space $U \coloneqq \{g \in C(\operatorname{dom}(T)) \mid g(0) = 0\}$. Given any $0 < \varepsilon \in \operatorname{dom}(T)$ such that $-\varepsilon \in \operatorname{dom}(T)$ and any $\delta > 0$. There exists a non-negative function $g \in U$ such that $(g - T) \restriction (\operatorname{dom}(T) \setminus (-\varepsilon, \varepsilon)) = 0$, $\|g\|_\infty = 1$. Thus there is a polynomial function p on $\operatorname{dom}(T)$ with constant term 0 such that $\|p\|_\infty < 1 + \delta$, $p(x) > -\delta$ for all $x \in \operatorname{dom}(T)$, $\|\,(p - T) \restriction (\operatorname{dom}(T) \setminus (-\varepsilon, \varepsilon))\,\|_\infty < \delta$. We can choose ε such that $|A_{-\varepsilon,\varepsilon}| < \delta$ according to lemma 12 (because the sets $A_{-\varepsilon,\varepsilon}$ are contained in $\mathring{\operatorname{supp}}(f)$, thus their intersection is empty). Now we get the estimate

$$\int_G |(E - (p \circ f))(x)| \,\mathrm{d}x = \int_G |T(f(x)) - p(f(x))| \,\mathrm{d}x$$

$$\leq \int_{\mathrm{supp}(f)\smallsetminus A_{-\varepsilon,\varepsilon}} \delta \, \mathrm{d}x + \int_{A_{-\varepsilon,\varepsilon}} (1+\delta) \, \mathrm{d}x \leq |\mathrm{supp}(f)| \, \delta + \delta \cdot (1+\delta),$$

which can become arbitrarily small. Thus, in L^1, we can approximate E by functions $p \circ f$ where p is a polynomial function vanishing at 0.

Thus, for some $n \in \mathbb{N}$ the vector $\pi(f^n)v$ is not equal to zero. In the compact case we get density of $V(f)$ in $C(G)$ by theorem 18, in the abelian case we get density in $L^1(G)$ by Wiener's theorem (17).

It remains to prove density of $V_A(f)$ in the case of a compact group G: In that case we can consider $[0,1]$ as the domain of T and choose (using the Müntz-Szasz theorem, 20) an approximating polynomial function p as before satisfying the additional constraint that all occuring exponents are elements of A. □

Let us restate two corollaries by Kerman and Weit for the general compact case:

Corollary 23. *Let G be compact. Given $f \in C(G)$. Assume that there is a unique $x_0 \in G$ such that $f(x_0) = \max_{x\in G} f(x)$. If $f(x_0) \neq 0$ or G is not discrete, then $V(f)$ is dense in $C(G)$. $V_0(f)$ is dense in $C(G)$ even if $f(x_0) = 0$ and G is discrete. The analogous results holds if the minimum is attained at a unique point (consider $-f$ instead of f) ([1], corollary 1).*

Proof. Let $f(x_0) = \max_{x\in G} f(x) =: \lambda$. For every $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$ there exists $u \in \mathcal{H}_\pi$ such that $\langle \pi(x_0)v, u\rangle > 0$. Since π is continuous, there exists an open neighbourhood U of x_0 such that $\langle \pi(x)v, u\rangle > 0$ for all $x \in U$. $\mathrm{supp}(f) \smallsetminus U$ is compact, choose $m := \max_{x\in\mathrm{supp}(f)\smallsetminus U} f(x)$ if $\mathrm{supp}(f) \smallsetminus U$ is inhabited and $m := \lambda - 1$ otherwise. We get $m < \lambda$ because the maximum of f only gets attained at x_0. Now choose

$$r = \begin{cases} \frac{m+\lambda}{2} & \text{for } 0 < m < \lambda \text{ or } m < \lambda < 0 \\ \frac{\lambda}{2} & \text{for } m \leq 0 < \lambda \end{cases}.$$

If $\lambda \neq 0$, then A_r contains an open neighbourhood of x_0. If $f(x_0) = 0$ and G is not discrete, then A_r is the open set $A_{m,-m}$, which is inhabited since x_0 is not isolated. In these cases we get $|A_r| > 0$. Since $A_r \subset U$ and since f is either always positive or always negative on A_r, $|\langle \pi(\chi_{A_r})v, u\rangle| = \left|\int_{A_r} \langle \pi(x)v, u\rangle \, \mathrm{d}x\right| > 0$. Now we have proved that for every $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$ there exists $r \in \mathbb{R}$ such that $\pi(\chi_{A_r})v \neq 0$. $V(f)$ is dense in $C(G)$ by theorem 21. If we want to consider $V_0(f)$, then we can always assume that f is positive implying $f(x_0) \neq 0$. □

When dropping discreteness or the non-zeroness of the maximum we get the following counter-example:

Example 3. Let G be the trivial group and $f = 0$. f attains a maximum at a unique point. $V_0(f)$ is dense in $C(G)$, but $V(f)$ is not.

Corollary 24. *Let G be compact. If for every $\pi \in \hat{G}, 0 \neq v \in \mathcal{H}_\pi$ there exists $0 \neq \lambda \in \mathbb{R}$ such that $\pi(\chi_{f^{-1}\{\lambda\}})v \neq 0$, then $V(f)$ is dense in $C(G)$.*

Proof. If $V(f)$ would not be dense in $C(G)$, there would exist $\pi \in \hat{G}, v \in \mathcal{H}_\pi$ such that for all $\lambda \in \mathbb{R}$ $\pi(\chi_{A_\lambda})v = 0$. Lemma 19 implies $\pi(\chi_{f^{-1}\{\lambda\}})v = 0$ ([1], corollary 2). $\square$

We notice that the characterisation in the abelian case does not work in $C_0(G)$ or $L^p(G)$ for $1 < p < \infty$:

Example 4. Consider the free discrete group $\mathbb{Z}$ and $f = \chi_{\{0,1\}}$. $V(f)$ is not dense in $L^1(\mathbb{Z})$, but it is dense in $C_0(\mathbb{Z})$ and $L^p(\mathbb{Z})$ for $1 < p < \infty$.

Proof. For $\lambda \in [0,1]$ $\chi_{A_\lambda} = f$ and the Fourier transform $\hat{f}\left(\frac{1}{2}\right) = \exp(0 \cdot 2\pi \mathrm{i}) + \exp\left(\frac{1}{2} \cdot 2\pi \mathrm{i}\right) = 0$ vanishes. By theorem 22 $V(f)$ is not dense in $L^1(\mathbb{Z})$. However, we can approximate the Dirac function $\delta = \chi_{\{0\}}$ uniformly using the sequence given by

$$\varphi_n := \sum_{0 \le i < n} (-1)^i \frac{n-i}{n} L_i f = 1 + \frac{1}{n} \sum_{0 \le i < n} (-1)^i \chi_{\{i+1\}}$$

for $n > 0$ because $\|\varphi_n - \delta\|_\infty = \frac{1}{n}$ and $\|\varphi_n - \delta\|_{L_p}^p = n \cdot \frac{1}{n^p} = n^{1-p}$ (converging for $p > 1$). $\square$

4 Functions attaining a maximum at a unique point

The following lemma, proved by Ray and Sarkar in [3], lemma 2.1.1., restated in [2], lemma 1, will be essential for characterising density of $V(f)$ in the case of a function with a unique maximum.

Lemma 25. *Let X be a locally compact space with a quasiregular (inner regular on open sets and outer regular) non-atomic Borel measure μ such that $\mu(U) > 0$ for every inhabited open set U. Consider a function $f \in C_c(X)$ such that there exists a unique point x_0, $f(x_0) = \max_{x \in X} f(x) > 0$. Then, for every $\psi \in C(X, \mathbb{C})$ such that $\psi(x_0) \neq 0$, there exists $0 < n \in \mathbb{N}$ such that $\int_X f^n(x)\psi(x)\,\mathrm{d}\mu(x) \neq 0$.*

Proof. We normalise f such that $f(x_0) = 1$. By multiplication with $\overline{\psi(x_0)}$ we may assume that $\Re\psi(x_0) > 0$. Then it suffices to prove that $\int_X f^n(x)\Re\psi(x)\,\mathrm{d}\mu(x) \neq 0$, thus assume without loss of generality that ψ is real-valued.

Now consider any $0 < \eta < 1$. We will prove that there exists a λ, $\eta < \lambda < 1$, such that $\int_{A_\lambda} \psi(x)\,\mathrm{d}x \neq 0$. Define $N := A_{\eta,\infty} \cap \psi^{-1}((0,\infty))$ which is an open neighbourhood of x_0 contained in $\mathrm{supp}(f)$. $\mathrm{supp}(f) \setminus N$ is compact, thus we can define $\lambda_1 := \max_{\mathrm{supp}(f) \setminus N}$, $0 < \lambda_1 < 1$ since $x_0 \in N$. Choose $1 > \lambda > \max\{\eta, \lambda_1\}$. We notice that $A_\lambda \subset N$. The open set $A_{\lambda,\infty}$ contains x_0, thus $\mu(A_\lambda) \ge \mu(A_{\lambda,\infty}) > 0$ implying $q := \int_{A_\lambda} \psi(x)\,\mathrm{d}x > 0$ since ψ is positive on N.

Define $b := \max_{x \in \mathrm{supp}(f)} |\psi(x)|$. Using lemma 12 we can choose $0 < \delta < \lambda$ such that $\mu(A_{\lambda-\delta,\lambda}) < \frac{q}{2b}$. Now choose a continuous (for example piecewisely linear), non-negative function $T \in C([\min_{x \in X} f, 1])$ such that $\mathrm{supp}(T) = [\lambda-\delta, 1]$, $\|T\|_\infty = 1$ and $T \restriction [\lambda, 1] = 1$. We get

$$\int_X (T \circ f)(x)\psi(x)\,\mathrm{d}x = \int_{A_{\lambda-\delta,\lambda}} (T \circ f)(x)\psi(x)\,\mathrm{d}x + \int_{A_\lambda} (T \circ f)(x)\psi(x)\,\mathrm{d}x$$

where we can estimate

$$\left|\int_{A_{\lambda-\delta,\lambda}}(T\circ f)(x)\psi(x)\,\mathrm{d}x\right| \leq \frac{q}{2b}\cdot b = \frac{q}{2}.$$

The second summand evaluates to

$$\int_{A_\lambda}(T\circ f)(x)\psi(x)\,\mathrm{d}x = \int_{A_\lambda}\psi(x)\,\mathrm{d}x = q.$$

This implies $\int_X (T\circ f)(x)\psi(x)\,\mathrm{d}x > \frac{q}{2}$.

Since $T(0)=0$, T can be uniformly approximated by a sequence of polynomial functions $(P_n)_{n\in\mathbb{N}}$ where the constant term $P_n(0)=0$ for all $n\in\mathbb{N}$. Thus $(P_n\circ f)_{n\in\mathbb{N}}$ uniformly approximates $T\circ f$ and $((P_n\circ f)\cdot\psi)_{n\in\mathbb{N}}$ uniformly approximates $(T\circ f)\cdot\psi$. Since all the functions are supported in $\operatorname{supp}(f)$, this implies that $((P_n\circ f)\cdot\psi)_{n\in\mathbb{N}}$ also converges to $(T\circ f)\cdot\psi$ in L^1 and by continuity of integration on L^1 the assumption that for all $0<n\in\mathbb{N}$ $\int_X f^n\cdot\psi = 0$ would imply $\int_X (T\circ f)\cdot\psi = 0$, a contradiction. Thus there exists $0<n\in\mathbb{N}$ such that $\int_X f^n(x)\psi(x)\,\mathrm{d}x \neq 0$. □

Notice that in the non-compact case $\mathfrak{C}\operatorname{supp}(f)$ contains at least two points, thus by uniqueness of the maximum we automatically get $f(x_0)>0$.

Lemma 26. *Let G_0 be an open subgroup of a locally compact group G and H a closed subgroup of G_0. For every $1\leq p<\infty$ and any dense subspace V of $L^p(G_0/H)$ the subspace $W\subset L^p(G/H)$ generated by all left-translates of elements of V is dense in $L^p(G/H)$.*

Proof. First we note that G_0 is a locally compact group since it is closed in G. Since G_0 is open, G_0 has positive measure with respect to the Haar measure on G. By uniqueness of the Haar measure $L^p(G_0/H)$ can be regarded as the subspace of all functions in $L^p(G/H)$ whose supports are contained in G_0/H.

Consider $f\in L^p(G/H)$. f vanishes outside of a σ-compact set, thus $\mathring{\operatorname{supp}}(f)$ is contained in a countable union of left cosets of G_0: The map $p\colon \mathring{\operatorname{supp}}(f)\to G/G_0$ mapping every point in $\mathring{\operatorname{supp}}(f)$ to its coset is continuous with respect to the discrete topology on G/G_0. Thus $p(\mathring{\operatorname{supp}}(f))$ is σ-compact, since $\mathring{\operatorname{supp}}(f)$ is σ-compact, in other words, $p(\mathring{\operatorname{supp}}(f))$ is at most countable. Given any $\varepsilon>0$. There exist finitely many cosets $g_1G_0,\dots g_nG_0$ such that $\|f-f\restriction(\{g_1,\dots,g_n\}(G_0/H))\|_{L^p}<\frac{\varepsilon}{2}$. For every g_i choose an approximation $a_i\in V$ such that

$$\left\|a_i-\Big((L_{g_i^{-1}}f)\restriction(G_0/H)\Big)\right\|_{L^p}<\frac{\varepsilon}{2n}.$$

We get an ε-approximation of f in W given by $\sum_{1\leq i\leq n}L_{g_i}a_i$. □

This lemma was applied by Weit to prove the following theorem ([2], theorem 1, generalised to homogeneous spaces by Ray and Sarkar in [4], theorem B):

Theorem 27. *Given a locally compact group G, a compact subgroup H and a function $f\in C_c(G/H)$ attaining a unique, positive maximum at the point x_0. $V(f)$ is dense in $L^p(G/H)$ for $1\leq p<\infty$ and in $C(G/H)$.*

Proof. Without loss of generality assume that the maximum is attained at eH. Define $\mathfrak{P}: C_c(G) \to C_c(G/H), (\mathfrak{P}f)(xH) := \int_H f(xh)\,\mathrm{d}h$ and $\mathfrak{T}: \mathbb{C}^{G/H} \to \mathbb{C}^G, (\mathfrak{T}f)(x) := f(xH)$.

For the L^p-case consider an open, σ-compact subgroup $G_0 < G$ containing $\mathrm{supp}(f)H$. We will prove that $V(f \restriction G_0/H)$ is dense in $L^p(G_0/H)$ implying density of $V(f)$ in $L^p(G/H)$ by lemma 26. Define $g := f \restriction G_0/H$. Assume that $V(g)$ is not dense in $L^p(G_0/H)$. Then there exists $0 \neq m \in L^q(G_0/H)$ ($\frac{1}{p} + \frac{1}{q} = 1$) annihilating—regarded as a functional on $L^p(G_0/H)$—all left translates of powers of g. Since $C_c(G_0)$ is dense in $L^p(G_0)$ and $M := \mathfrak{T}m \neq 0$ there exists $\varphi \in C_c(G_0)$ such that $(\Delta_H^{-1} \cdot \check{M}) * \varphi \neq 0$ (since $((\Delta_H^{-1} \cdot \check{M}) * \varphi)(e) = \int M\varphi$, $(\Delta_{G_0}^{-1} \cdot \check{M}) * \varphi$ is well-defined since Δ_{G_0} is bounded on compact sets). $(\Delta_{G_0}^{-1} \cdot \check{M}) * \varphi$ is continuous and has the same non-zero value at every point in H, thus $\mathfrak{P}((\Delta_{G_0}^{-1} \cdot \check{M}) * \varphi)(eH) \neq 0$. We apply lemma 25 to $\mathfrak{P}(\check{\mathfrak{T}f})$, which attains its maximum at eH and get for some $n > 0$ (using Fubini-Tonelli, which is applicable since G_0 is σ-compact and all integrands are compactly supported and bounded)

$$\begin{aligned}
0 \neq & \int_{G_0/H} (\mathfrak{P}((\Delta_{G_0}^{-1} \cdot \check{M}) * \varphi))(xH) \mathfrak{P}(\check{\mathfrak{T}f})^n (xH)\,\mathrm{d}xH \\
= & \int_{G_0/H} \int_H ((\Delta_{G_0}^{-1} \cdot \check{M}) * \varphi)(xh)\,\mathrm{d}h \cdot \int_H (\check{\mathfrak{T}f})^n (xh')\,\mathrm{d}h'\,\mathrm{d}xH \\
= & \int_{G_0/H} \int_H \int_{G_0} \Delta_{G_0}^{-1}(y) \check{M}(y) \varphi(y^{-1}xh)\,\mathrm{d}y\,\mathrm{d}h \cdot \int_H (\check{\mathfrak{T}f})^n (xh')\,\mathrm{d}h'\,\mathrm{d}xH \\
= & \int_{G_0} \int_{G_0} \int_H \Delta_{G_0}^{-1}(y) \check{M}(y) \varphi(y^{-1}xh)\,\mathrm{d}h \cdot (\check{\mathfrak{T}f})^n (x)\,\mathrm{d}x\,\mathrm{d}y \\
= & \int_{G_0} \int_{G_0} \int_H \Delta_{G_0}^{-1}(y) \check{M}(y) \varphi(xh)\,\mathrm{d}h \cdot (\check{\mathfrak{T}f})^n (yx)\,\mathrm{d}x\,\mathrm{d}y \\
= & \int_{G_0} \int_H \varphi(xh)\,\mathrm{d}h \cdot \int_{G_0} \Delta_{G_0}^{-1}(y) \check{M}(y) (\check{\mathfrak{T}f})^n (yx)\,\mathrm{d}y\,\mathrm{d}x \\
= & \int_{G_0} \int_H \varphi(xh)\,\mathrm{d}h \cdot \int_{G_0} \Delta_{G_0}^{-1}(y) M(y^{-1}) (\mathfrak{T}f)^n (x^{-1}y^{-1})\,\mathrm{d}y\,\mathrm{d}x \\
= & \int_{G_0} \int_H \varphi(xh)\,\mathrm{d}h \cdot \int_{G_0} M(y) (\mathfrak{T}f)^n (x^{-1}y)\,\mathrm{d}y\,\mathrm{d}x \\
= & \int_{G_0} \int_H \varphi(xh)\,\mathrm{d}h \cdot \left(\int_{G_0} M L_x \mathfrak{T} f^n\right) \mathrm{d}x \\
= & \int_{G_0} \int_H \varphi(xh)\,\mathrm{d}h \cdot \left(\int_H \int_{G_0/H} m L_x f^n\right) = 0,
\end{aligned}$$

a contradiction.

Notice that the transition to the σ-compact subgroup in the proof was actually only necessary for the case $p = 1$.

Assume that $V(f)$ is not dense in $C(G/H)$. Thus there exists (using Hahn-Banach for locally convex spaces) a non-zero $\mu \in C(G/H)'$ (a compactly supported measure) annihilating all $L_x f^n$. Define $\nu := \mu \circ \mathfrak{P}$. Since $C_c(G)$ is dense in $C(G)$ and $\nu \neq 0$, there exists $\varphi \in C_c(G)$ such that $\tilde{\nu} * \varphi \neq 0$ (since $(\tilde{\nu} * \varphi)(e) = \nu(\varphi)$). $\tilde{\nu} * \varphi$ is compactly supported, continuous and has the same non-zero value at every point in H, thus $\mathfrak{P}(\check{\nu} * \varphi)(eH) \neq 0$. We apply lemma 25 to $\mathfrak{P}(\check{\mathfrak{T}f})$, which attains its maximum at eH and get for some $n > 0$ (using Fubini-Tonelli, which is applicable since all occuring integrals exist and all occuring supports are contained in a compact set)

$$\begin{aligned}
0 \neq & \int_{G/H} \mathfrak{P}(\tilde{\nu} * \varphi)(xH) \cdot \mathfrak{P}(\check{\mathfrak{T}f})^n (xH)\,\mathrm{d}xH \\
= & \int_{G/H} \int_H (\tilde{\nu} * \varphi)(xh)\,\mathrm{d}h \cdot \int_H (\check{\mathfrak{T}f})^n (xh')\,\mathrm{d}h'\,\mathrm{d}xH
\end{aligned}$$

$$
\begin{aligned}
&= \int_{G/H}\int_H\int_G \varphi(y^{-1}xh)\,\mathrm{d}\tilde{\nu}(y)\,\mathrm{d}h\cdot\int_H \check{(\mathfrak{T}f)}^n(xh')\,\mathrm{d}h'\,\mathrm{d}xH\\
&= \int_G\int_G\int_H \varphi(yxh)\,\mathrm{d}h\cdot\check{(\mathfrak{T}f)}^n(x)\,\mathrm{d}x\,\mathrm{d}\nu(y)\\
&= \int_G\int_G\int_H \varphi(xh)\,\mathrm{d}h\cdot\check{(\mathfrak{T}f)}^n(y^{-1}x)\,\mathrm{d}x\,\mathrm{d}\nu(y)\\
&= \int_G\int_H \varphi(xh)\,\mathrm{d}h\cdot\int_G(L_y\check{(\mathfrak{T}f)}^n)(x)\,\mathrm{d}\nu(y)\,\mathrm{d}x\\
&= \int_G\int_H \varphi(xh)(\nu*\check{(\mathfrak{T}f)}^n)(x)\,\mathrm{d}x\\
&= \int_G\int_H \varphi(xh)\,\mathrm{d}h\cdot\tilde{\nu}(R_x\check{(\mathfrak{T}f)}^n)\,\mathrm{d}x\\
&= \int_G\int_H \varphi(xh)\,\mathrm{d}h\cdot\nu(L_x(\mathfrak{T}f)^n)\,\mathrm{d}x\\
&= \int_G\int_H \varphi(xh)\,\mathrm{d}h\cdot\int_{G/H}\int_H L_x(\mathfrak{T}f)^n(yh')\,\mathrm{d}h'\,\mathrm{d}\nu(yH)\,\mathrm{d}x\\
&= \int_G\int_H \varphi(xh)\,\mathrm{d}h\cdot\int_H\int_{G/H} L_xf^n(yH)\,\mathrm{d}\nu(yH)\,\mathrm{d}h'\,\mathrm{d}x = 0.
\end{aligned}
$$

This is a contradiction. □

Notice that we already know this theorem for compact groups.

Corollary 28. *Let G be a compact group. For density of $V_0(f)$ in $C(G)$ it is sufficient that there is a value of f which is only attained at a single point x_0.*

Proof. Consider the function $-(f-f(x_0))^2$, which has a unique maximum at x_0. □

5 The algebra generated by the translates of a function

In this section G will denote a locally compact group.

Lemma 29. *Given $f \in C(G)$. There exists a subgroup $S_f^r < G$ containing exactly those $x \in G$ such that $R_x f = f$. S_f^r is closed.*

Proof. S_f^r is the stabilizer subgroup of f with respect to right-translations, we see that

$$S_f^r = \bigcap_{x\in G}\{y \in G \mid f(xy) = f(x)\}.$$

S_f^r is the intersection of a family of continuous preimages of one-point-subsets of $\mathbb{C}$, thus it is closed. □

Lemma 30. *For $f \in C_c(G)$, S_f^r is compact.*

Proof. $S_f^r \subset \operatorname{supp}(f)^{-1}\operatorname{supp}(f)$ and S_f^r is closed, thus S_f^r is compact. □

The characterisation of density of $A(f)$ we will prove is a direct consequence of the Stone-Weierstraß theorem:

Theorem 31. *Let X be a locally compact space. A closed subalgebra $A \subset C_0(X)$ is equal to $C_0(X)$ if and only if the following conditions hold:*

- *A separates points.*

- *For every point $x \in X$ there exist functions $f_0, f_1 \in A$ such that $f_0(x) \neq f_1(x)$.*

Theorem 32. *Consider a non-trivial locally compact group G and $f \in C_0(G)$. $A(f)$ is dense in $C_0(G)$ if and only if S_f^r is trivial.*

Proof. First assume that S_f^r is non-trivial. There exists $x \in S_f^r \setminus \{e\}$. Now for any $y \in G$ we get $(L_y f)(e) = (L_y R_x f)(e) = f(y^{-1}x) = (L_y f)(x)$. Thus $A(f)$ does not separate points. It is not dense in $C_0(G)$ according to the Stone-Weierstraß theorem (31).

For the opposite direction assume that $A(f)$ is not dense in $C_0(G)$. If there is a point $x \in G$ such that $(L_y f)(x) = 0$ for all $y \in G$, then obviously $f = 0$, thus $S_f^r = G$. Otherwise there exist $x \neq y \in G$ such that for all $z \in G$ we get $(L_z f)(x) = (L_z f)(y)$. Now consider any $g \in G$, we insert $z = xg^{-1}$ and get $(L_{xg^{-1}} f)(x) = (L_{xg^{-1}} f)(y)$, thus $f(g) = f(gx^{-1}y)$ implying $R_{x^{-1}y} f = f$. We get $e \neq x^{-1}y \in S_f^r$. □

This theorem also provides a sufficient condition for density of $V(f)$ assuming that $V(f)$ is dense in $A(f)$. The latter condition can be slightly weakened using the following lemma.

Lemma 33. *If every function $f L_x f^n$ for $x \in G, n \in \mathbb{N}$ is an element of $\overline{V(f)}$, then $A(f) \subset \overline{V(f)}$ (with respect to the uniform norm).*

Proof. It suffices to prove that every product $L_{x_1} f^{n_1} \cdot \ldots \cdot L_{x_m} f^{n_m}$ can be uniformly approximated by functions in $V(f)$. We will prove the lemma by induction with respect to m. The case $m = 1$ is trivial ($L_{x_1} f^{n_1} \in V(f)$). Assume that the claim holds for a $m \geq 1$. Consider a function $F := L_{x_1} f^{n_1} \cdot \ldots \cdot L_{x_m} f^{n_m} \cdot L_{x_{m+1}} f^{n_{m+1}}$. Without loss of generality we can assume that $x_{m+1} = e$ (otherwise consider a translate) and $n_{m+1} = 1$ (the general case follows by induction). Consider $\varepsilon > 0$. There exists an ε-approximation $L_{y_1} f^{r_1} + \ldots + L_{y_l} f^{r_l}$ of F. For each $f L_{y_i} f^{r_i}$ $(1 \leq i \leq l)$ there exists an $\frac{\varepsilon}{l}$-approximation $g_i \in V(f)$. Now $g_1 + \ldots + g_l$ is a 2ε-approximation of F: $\|F - g_1 - \ldots - g_l\|_\infty \leq \varepsilon + \|F - (L_{y_1} f^{r_1} + \ldots + L_{y_l} f^{r_l})\|_\infty \leq 2\varepsilon$. □

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