Model theoretic properties of logics with team semantics

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The idea of team semantics has been introduced by Hodges to provide compositional semantics for independence friendly logic. Väänänen used this approach to define dependence logic which appears to be more natural in the context of team semantics. Since then various other logics with team semantics have been considered in the literature. We will give an overview over some key concepts and then prove various specific results for such logics.

We will always assume ZFC as a set theoretic foundation (at some points we will give meta-mathematical statements about ZFC, but the meta-language will require only a very weak foundation). Some particular properties of ZF will play an important role in analysing some degrees of uncomputability. The axiom of choice will be used to argue using infinite paths carelessly, choosing winning strategies and for compactness arguments. All structures will be relational first-order structures.

1 Notation

An *occurence of a subformula* of φ is defined as a triple consisting of φ and a pair of indices denoting the beginning and the end of a subformula inside φ. Given two occurences of subformulas ψ, χ of φ: If φ has a $\vee$ at position j and $\psi = (\varphi, i, j)$ and $\chi = (\varphi, j+1, k)$, then the expression $\varphi \vee \psi$ will denote the occurence of a subformula starting at the beginning of φ and ending at the end of ψ (namely (φ, i, k)). Other syntactic operations on occurences of subformulas are defined analogously. We identify φ and $(\varphi, 0, |\varphi|)$. Every occurence of a subformula can be regarded as a subformula (but not vice-versa), thus we will apply all notions for subformulas to occurences of subformulas, too.

For every formula φ $fr(\varphi)$ denotes the set of all free variables in φ. We will always assume that for every subformula ψ of φ and every subformula χ of ψ $fr(\psi) \subset fr(\chi)$. Given some fixed set of variables v containing all the free variables of a formula φ. For occurences of subformulas ψ of φ we define $var_v(\psi)$ as follows:

- $var_v(\varphi) = v$
- If $\psi = \chi \vee \eta$ or $\psi = \chi \wedge \eta$, then $var_v(\chi) = var_v(\eta) = var_v(\psi)$.
- If $\psi = \neg\chi$, then $var_v(\chi) = var_v(\psi)$.
- If $\psi = \exists x\chi$ or $\psi = \forall x\chi$, then $var_v(\chi) = var_v(\psi) \cup \{x\}$.

Usually v is clear from the context and we will not write it down (usually $v = fr(\varphi)$).

2 Team semantics

Definition 1. A *logic with team semantics* consists of

- a class of *atomic formulas* containing all formulas $R\bar{x}$ given a relation symbol R,

- a set $fr(\varphi)$ for every atomic formula φ and
- team semantics for atomic formulas, i. e. a set $[\![\varphi]\!]_v^{\mathfrak{A}} \subset \mathcal{P}(A^v)$ for every atomic or negated atomic formula φ, every first-order structure $\mathfrak{A}$ with a universe A and every set v containing the free variables in φ. For a relation symbol R in the signature of A we require $[\![R\bar{x}]\!]_v^{\mathfrak{A}} = \{X \subset A^v \mid \text{for all } \bar{a} \in X : \mathfrak{A}, \bar{a} \models R\bar{x}\}$ and $[\![\neg R\bar{x}]\!]_v^{\mathfrak{A}} = \{X \subset A^v \mid \text{for all } \bar{a} \in X : \mathfrak{A}, \bar{a} \models \neg R\bar{x}\}$.

The formulas of the logic are given by combinations of atomic formulas using $\forall, \exists, \wedge, \vee, \neg$. Sets of free variables for arbitrary formulas are defined in the obvious way. We will always assume that variables do not get used multiple times. Semantics will be defined using the negation normal form: We define that every formula which is not in negation normal form is equivalent to a negation normal form which gets constructed as in first-order logic ($\neg\forall \rightsquigarrow \exists\neg$, $\neg\exists \rightsquigarrow \forall\neg$, $\neg(\cdot \wedge \cdot) \rightsquigarrow \neg \cdot \vee \neg \cdot$, $\neg(\cdot \vee \cdot) \rightsquigarrow \neg \cdot \wedge \neg \cdot$).

Given a structure $\mathfrak{A}$, a formula φ in such a logic, and a *team* $X \subset A^v$ (we will always call such sets teams) where $fr(\varphi) \subset v$. We define $\mathfrak{A} \models_X \varphi$ inductively:

- If φ is atomic or negated atomic, then $\mathfrak{A} \models_X \varphi$ if and only if $X \in [\![\varphi]\!]_v^{\mathfrak{A}}$.
- If $\varphi = \psi \vee \theta$, then $\mathfrak{A} \models_X \varphi$ if and only if there exist teams $Y, Z \subset A^v$ such that $X = Y \cup Z$, $\mathfrak{A} \models_Y \psi$, and $\mathfrak{A} \models_Z \theta$.
- If $\varphi = \psi \wedge \theta$, then $\mathfrak{A} \models_X \varphi$ if and only if $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \theta$.
- If $\varphi = \exists x\psi$, then $\mathfrak{A} \models_X \varphi$ if and only if there exists a team $Y \subset A^{v \cup \{x\}}$ such that $\mathfrak{A} \models_Y \psi$ and $X = Y \restriction v := \{s \restriction v \mid s \in Y\}$.
- If $\varphi = \forall x\psi$, then $\mathfrak{A} \models_X \varphi$ if and only if $\mathfrak{A} \models_{X(A/x)} \psi$ where $X(A/x)$ denotes the set of all assignments in $A^{v \cup \{x\}}$ such that their restriction to v is an element of X.

For a general formula φ $[\![\varphi]\!]_v^{\mathfrak{A}}$ denotes the set of all teams $X \subset A^v$ such that $\mathfrak{A} \models_X \varphi$. We might omit the index v to denote $[\![\varphi]\!]_{fr(\varphi)}^{\mathfrak{A}}$.

As long as we do not want to extend our logic by additional connectives we can give an equivalent definition of satisfaction:

Definition 2. Given a structure $\mathfrak{A}$. A team $X \subset A^v$ where $fr(\varphi) \subset v$ satisfies a formula φ ($\mathfrak{A} \models_X \varphi$, $X \in [\![\varphi]\!]_v^{\mathfrak{A}}$) if and only if it satisfies the negation normal form of φ. Assume that φ is in negation normal form. Then $\mathfrak{A} \models_X \varphi$ if and only if there exists a *witnessing family of teams* (X_ψ) for $\mathfrak{A}, X, \varphi$ (teams indexed by all occurences of subformulas ψ of φ except of the occurences of atomic formulas inside a negated atomic formula), i. e.:

- $X_\varphi = X$
- $X_\psi \in [\![\psi]\!]_{var_v(\psi)}^{\mathfrak{A}}$ for every atomic or negated atomic occurence of a subformula ψ
- $X_{\psi \vee \eta} = X_\psi \cup X_\eta$ for every occurence of a subformula $\psi \vee \eta$
- $X_{\psi \wedge \eta} = X_\psi = X_\eta$ for every occurence of a subformula $\psi \wedge \eta$

- $X_{\exists x\psi} = X_\psi \upharpoonright var(\exists x\psi)$ for every occurence of a subformula $\exists x\psi$
- $X_{\forall x\psi}(A/x) = X_\psi$ for every occurence of a subformula $\forall x\psi$

Theorem 1. *If the semantics of atomic and negated atomic formulas are defined by first-order formulas (i. e. $[\![\alpha]\!]^{\mathfrak{A}}_v = \{X \subset A^v \mid X \circ \bar{x} \vDash \psi\}$ for some enumeration $\bar{x}$ of $fr(\alpha)$ and some first-order sentence ψ for all atomic and negated atomic formulas α), then this logic can be translated into Σ^1_1 (i. e. for every φ in the logic over a signature τ and any finite set of variables $v \supset fr(\varphi)$ there is a Σ^1_1-formula φ' over the signature $\tau \cup \{R\}$ where R is a $|v|$-ary relation symbol such that $\mathfrak{A}, R \mapsto X \vDash \varphi'$ (where X gets interpreted as a set of tuples) if and only of $\mathfrak{A} \vDash_X \varphi$).*

Proof. Immediate, since the existence of a witnessing family of teams is a Σ^1_1-property. □

Definition 3. A logic with team semantics is said to *respect locality* if and only if for every formula φ, every model $\mathfrak{A}$, and every team $X \subset A^v$ where $fr(\varphi) \subset v$ $\mathfrak{A} \vDash_X \varphi$ is equivalent to $\mathfrak{A} \vDash_{X \upharpoonright fr(\varphi)} \varphi$.

Lemma 2. *If the semantics of atomic and negated atomic formulas respect locality, then the whole logic respects locality assuming that we only use the connectives mentioned above.*

Proof. Immediate by induction on formulas. □

Now we will define the logics with team semantics we will consider:

Definition 4. We define the following atomic formulas:

- Dependence: $\mathfrak{A} \vDash_X =\!(\bar{x}, y)$ if and only if the value of y in X only depends on the value of $\bar{x}$, i. e. for two assignments $s, t \in X$ such that $[\![\bar{x}]\!]^s$ (which can formally be defined as $s \circ \bar{x}$) and $[\![\bar{x}]\!]^t$ agree we also have $s(y) = t(y)$.
- Independence: $\mathcal{A} \vDash_X \bar{x} \perp_{\bar{z}} \bar{y}$ if and only if the values of $\bar{x}$ and $\bar{y}$ are independent given a specific value of $\bar{z}$, i. e. for every subteam $Y \subset X$ such that $\bar{z}$ is constant $[\![\bar{x}\bar{y}]\!]^Y = [\![\bar{x}]\!]^Y \times [\![\bar{y}]\!]^Y$ ($[\![\bar{x}]\!]^Y := \{[\![\bar{x}]\!]^s \mid s \in Y\}$).
- Inclusion: $\mathfrak{A} \vDash_X \bar{x} \subset \bar{y}$ where $\bar{x}, \bar{y}$ have the same length if and only if $[\![\bar{x}]\!]^Y \subset [\![\bar{y}]\!]^Y$.
- Exclusion: $\mathfrak{A} \vDash_X \bar{x} | \bar{y}$ where $\bar{x}, \bar{y}$ have the same length if and only if $[\![\bar{x}]\!]^Y \cap [\![\bar{y}]\!]^Y = \emptyset$.

Analogously to the remark on page 24 in [20] we define that the negations of such atomic formulas are only satisfied by the empty team.

We define the following logics with team semantics by using the connectives and quantifiers $\vee, \wedge, \neg, \exists, \forall$ and certain atoms:

- Constancy logic uses atoms of the form $=\!(x)$. On sentences (closed formulas) it is known to be equivalent to first-order logic ([6], corollary 3.14).

- Dependence logic $\mathcal{D}$ uses dependence atoms.
- Independence logic $\mathcal{I}$ uses independence atoms.
- Inclusion logic $\mathcal{INC}$ uses inclusion atoms.
- Exclusion logic uses exclusion atoms. It is known to be equivalent to dependence logic ([6], corollary 4.20).
- I/E-logic uses both inclusion and exclusion atoms. It is known to be equivalent to independence logic ([6], corollary 4.23).

We will now introduce some connectives, which turned out to enable logics with team semantics to express non-Σ_1^1-properties although they can be translated to Σ_1^1 as long as they do not use these connective:

Definition 5.

- Intuitionistic implication: $\mathfrak{A} \models_X \varphi \to \psi$ if and only if for all $Y \subset X$ $\mathfrak{A} \models_Y \varphi$ implies $\mathfrak{A} \models_Y \psi$ ([1],[7], p. 35).
- Maximal implication: $\mathfrak{A} \models_X \varphi \hookrightarrow \psi$ if and only if for all maximal $Y \subset X$ with the property $\mathfrak{A} \models_Y \varphi$ we have $\mathfrak{A} \models_Y \psi$ ([7], p. 178).
- Linear implication: $\mathfrak{A} \models_X \varphi \multimap \psi$ if and only if for all teams Y satisfying φ $\mathfrak{A}_{X \cup Y} \models \psi$ ([1], [7], p. 35).
- Classical negation: $\mathfrak{A} \models_X \ \varphi$ if and only if $\mathfrak{A} \not\models_X \varphi$ ([20], p. 144).

Dependence logic with intuitionistic implication is called *intuitionistic dependence logic*. Dependence logic with classical negation is called *team logic*.

3 Closure properties

The following property of dependence logic is fundamental:

Lemma 3. *The teams satisfying a $\mathcal{D}$-formula are closed under taking arbitrary subsets ([20], proposition 3.10).*

Proof. Every atomic or negated atomic formula of dependence logic is closed under taking arbitrary subsets, by induction this property gets transferred to $\mathcal{D}$. In fact the same argument works whenever the semantics of atomic and negated atomic formulas have this closure properties and we only use $\vee, \wedge, \forall, \exists, \neg, \to, \hookrightarrow, \multimap$ □

In fact $\mathcal{D}$ is equivalent to the fragment of Σ_1^1 closed under taking subsets.

Lemma 4. *If every atomic or negated atomic formula in a logic with team semantics only using $\vee, \wedge, \forall, \exists, \neg, \to, \hookrightarrow$ is satisfied by the empty team, then every formula is satisfied by the empty team (this holds for dependence, inclusion and independence logic).*

Proof. Immediate by induction. □

In fact $\mathcal{I}$ is equivalent to the fragment of Σ_1^1 providing satisfaction by the empty team ([6], theorem 6.2).

Lemma 5. *If for every atomic or negated atomic formula the satisfying teams are closed under unions, then this property holds for every formula in a logic only using $\vee, \wedge, \forall, \exists, \neg$. Especially this property holds for inclusion logic.*

Lemma 6. *The teams satisfying a $\mathcal{I}$-formula where for each occurence $\bar{x}\bot_{\bar{z}}\bar{y}$ the tuples $\bar{x}, \bar{y}$ only consist of free variables of the formula are closed under taking subsets of cardinality ≤ 1 (we will call such formulas* 1-closed*).*

Proof. Formulas of the type $\bar{x}\bot_{\bar{z}}\bar{y}$ are trivially satisfied if the values of $\bar{x}$ and $\bar{y}$ are unique. The property is transferred to the formulas as described in the lemma. □

Lemma 7. *If φ is* 1*-closed, $\exists x\varphi$ is also* 1*-closed (it does not matter whether we are using strict or lax semantics).*

Corollary 8. *It is impossible to translate inclusion logic to independence logic without introducing additional universal quantifiers.*

Proof. The formula $x \subset y$ does not satisfy the closure property from lemma 6. □

Lemma 9. *If φ is a first-order formula, then $\mathfrak{A} \vDash_X \varphi$ holds if and only if $\mathfrak{A} \vDash_{\{\bar{x}\}} \varphi$ for all $\bar{x} \in X$.*

Proof. This is a consequence of the fact that the teams satisfying a first-order formula are both closed under taking subsets and taking unions by the previous lemmas. □

4 Σ_1^1 translation and downwards closed formulas

The following remarks will be useful in the next section. We will use the following result characterising formulas of dependence logic:

Theorem 10. *A Σ_1^1 formula describing a team using the variable Y is equivalent to a formula of dependence logic if and only if Y appears only negatively ([12], theorem 4.10).*

Lemma 11. *For every formula $\varphi(Y)$ in Σ_1^1 describing teams (and every formula in a logic with team semantics translatable to Σ_1^1) there exists a formula $\varphi^{\downarrow} \in \mathcal{D}$ such that the teams satisfying $\varphi^{\downarrow}$ are exactly the subteams of teams satisfying φ.*

Proof. Define $\psi(X) := \exists Y\,((\forall x Xx \to Yx) \wedge \varphi(Y)) \in \Sigma_1^1$. X appears only negatively, thus it is equivalent to a formula of dependence logic, namely $\varphi^{\downarrow}$. □

Example 1. *A formula φ in a logic translatable to Σ_1^1 is equivalent to a formula of dependence logic if and only if $\varphi \equiv \varphi^{\downarrow}$.*

Lemma 12. *$\varphi \vDash \varphi^{\downarrow}$.*

Lemma 13. *For every Σ_1^1-formula φ and every $\psi \in \mathcal{D}$ the implication $\varphi \Rightarrow \psi$ holds if and only if $\varphi^{\downarrow} \Rightarrow \psi$ does.*

Proof. If $\varphi^{\downarrow} \vDash \psi$ then $\varphi \vDash \varphi^{\downarrow} \vDash \psi$ according to lemma 12. Now suppose that $\varphi \vDash \psi$ and $\mathfrak{A} \vDash_X \varphi^{\downarrow}$. There exists $Y \supset X$ such that $\mathfrak{A} \vDash_Y \varphi$. Thus $\mathfrak{A}_Y \vDash \psi$. Since the teams satisfying ψ are closed under taking subteams (lemma 3), $\mathfrak{A} \vDash_X \psi$. □

5 Negation

In [13] Kontinen and Väänänen proved a theorem guaranteeing that two formulas of dependence logic can be interpolated by a formula and its negation. We will try to generalise this theorem to other logics with team semantics. In this section the cardinality of all structures is assumed to be greater than one. We will start with a definition:

Definition 6. A logic with team semantics has a *totally undetermined sentence* $\perp\!\!\!\perp$ if $\perp\!\!\!\perp^{\mathfrak{A}} = \{\emptyset\} = \perp\!\!\!\perp^{\mathfrak{A}}$ holds for all structures $\mathfrak{A}$.

The existence of such a sentence depends on the definition of the semantics of the negation of the atomic formulas expressing dependence, independence, inclusion etc. For many results regarding logics with team semantics these negations are not that interesting and usually they get ignored. However, those semantics can be chosen in a natural way implying that totally undetermined sentences appear naturally—inducing a surprising property of negation.

Lemma 14. *Constancy logic $\mathcal{C}$ has a totally undetermined sentence.*

Proof. Choose $\perp\!\!\!\perp := \forall x =\!(x)$. □

Corollary 15. *Dependence und independence logic have totally undetermined sentences.*

Totally undetermined sentences empower us to strengthen an interpolation property (the proof is essentially the same as in [13], theorem 3.1):

Theorem 16. *Assume that a logic L (with team-semantics, containing FO) has a totally undetermined sentence $\perp\!\!\!\perp$. Then for $\varphi, \psi \in L$ these statements are equivalent:*

- *There exists $\eta \in L$ such that $\varphi \Rightarrow \eta$ and $\psi \Rightarrow \neg\eta$*
- *There exists $\theta \in L$ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$*

Proof. Consider φ, ψ, η as described above. Choose $\hat{\varphi} := \varphi \vee \perp\!\!\!\perp \equiv \varphi$, $\hat{\psi} := \psi \vee \perp\!\!\!\perp \equiv \psi$ and $\theta := \hat{\varphi} \wedge (\neg\hat{\psi} \vee \eta)$. We get:

$$\begin{aligned} \neg\hat{\varphi} &\equiv \neg\varphi \wedge \neg\perp\!\!\!\perp && \equiv \perp \\ \neg\hat{\psi} &\equiv \neg\psi \wedge \neg\perp\!\!\!\perp && \equiv \perp \\ \theta &\equiv \varphi \wedge (\perp \vee \eta) && \equiv \varphi \wedge \eta \equiv \varphi \\ \neg\theta &\equiv \neg\hat{\varphi} \vee (\hat{\psi} \wedge \neg\eta) && \equiv \perp \vee (\psi \wedge \neg\eta) \equiv \psi \end{aligned}$$

The converse is obvious: Choose $\eta = \theta$. □

Now we will analyse when this interpolation is possible. In [13] it has been proved for contradictory formulas of dependence logic.

Definition 7. Two formulas in a logic with team semantics are called *contradictory* if the empty team is the only team satisfying both formulas.

It will turn out that in general contradictoriness does not suffice, we will need a stronger condition:

Definition 8. Two formulas φ, ψ in a logic with team semantics are said to be *strongly contradictory* if $X \cap Y = \emptyset$ for all teams X, Y and structures $\mathfrak{A}$ such that $\mathfrak{A} \models_X \varphi$ and $\mathfrak{A} \models_Y \psi$.

Lemma 17. *If φ, ψ are 1-closed (this holds especially for sentences and for $\varphi, \psi \in \mathcal{D}$), then φ, ψ are contradictory if and only if they are strongly contradictory.*

Lemma 18. *φ, ψ are strongly contradictory if and only if $\varphi^{\downarrow}, \psi^{\downarrow}$ are contradictory.*

By using 18 we can reduce the case of general logics with team semantics to the case of dependence logic, but we will give a self-contained proof.

Theorem 19. *If φ, ψ satisfy the equivalent statements in theorem 16 and assume that every atomic formula of the logic is strongly contradictory to its negation (that is true for $\mathcal{D}$, $\mathcal{I}$ and inclusion logic), then φ, ψ are strongly contradictory.*

Proof. We will prove that *every* formula θ is strongly contradictory to its negation. Consider $\mathfrak{A} \models_X \theta$ and $\mathfrak{A} \models_Y \neg\theta$. We will prove $X \cap Y = \emptyset$ by induction on θ. Assume that θ is given in negation normal form and let θ' be the negation normal form of $\neg\theta$. Obviously $X \cap Y = \emptyset$ holds for atomic formulas (and in fact for any formula in first-order or dependence logic by using downward closedness). Assume that the theorem holds for α, β, then it also holds for:

- $\eta = \alpha \wedge \beta$: $\eta' = \alpha' \vee \beta'$. $\mathfrak{A} \models_X \alpha$ and $\mathfrak{A} \models_X \beta$. Choose $P \cup Q = Y$ such that $\mathfrak{A} \models_P \alpha'$ and $\mathfrak{A} \models_Q \beta'$. $P \cap X = \emptyset$ and $Q \cap X = \emptyset$ thus $Y \cap X = \emptyset$.
- $\eta = \alpha \vee \beta$: analogously.
- $\eta = \forall x \alpha$: $\eta' = \exists x \alpha'$. $\mathfrak{A} \models_{X(A/x)} \alpha$ and there exists a total relation R such that $\mathfrak{A} \models_{Y(R/x)} \alpha'$. $X(A/x) \cap Y(R/x) = \emptyset$ implies $X \cap Y = \emptyset$.
- $\eta = \exists x \alpha$: analogously.

Thus θ and $\neg\theta$ are strongly contradictory. Now choose $\theta \equiv \varphi$ and $\neg\theta \equiv \psi$, Thus φ and ψ are strongly contradictory. □

Example 2. *$\varphi := \forall y \; y \subset x$ and $\psi :=\,=\!(x)$ are contradictory but theorem 16 does not apply.*

Proof. Because independence logic respects locality, we only have to consider subsets of the universe as teams. The only team satisfying φ is the universe itself. The teams satisfying ψ are teams consisting of a single element. Since we require the universe to contain at least two elements, φ and ψ are contradictory. However, they are not strongly contradictory because the universe contains all subteams consisting of a single element. □

Theorem 20. *For logics with team semantics containing FO having a totally undetermined sentence and being translatable to Σ_1^1 strong contradictoriness of φ, ψ implies the existence of a formula θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (the converse of theorem 19 holds, the proof is analogous to the proof of theorem 3.1 in [13]).*

Proof. Let φ, ψ be strongly contradictory. Then $\varphi^{\downarrow}$ and $\psi^{\downarrow}$ are contradictory formulas of dependence logic, $\varphi \Rightarrow \varphi^{\downarrow}$ and $\psi \Rightarrow \psi^{\downarrow}$. Let $\exists R_1 \ldots \exists R_n \varphi^{\circ}, \exists S_1 \ldots \exists S_m \psi^{\circ}$ be Σ_1^1 translations of $\varphi^{\downarrow}, \psi^{\downarrow}$, where $\varphi^{\circ}, \psi^{\circ}$ use the unary predicate T to refer to the team. Without loss of generality assume $R_i \neq S_j$. By applying Craig's interpolation theorem for first-order logic to the contradictory first-order formulas $\varphi^{\circ}, \psi^{\circ} \wedge \exists \bar{y} T \bar{y}$, there exists $\gamma \in FO$ not containing the relations R_i, S_j such that $\varphi^{\circ} \Rightarrow \gamma$ and $\psi^{\circ} \wedge \exists \bar{y} T \bar{y} \Rightarrow \neg\gamma$. Define the first-order formula η by replacing all occurences of $T\bar{y}$ in γ by $\bar{x} = \bar{y}$ where x_i are the free variables in η. If $|X| = 1$ for a team X, then $\mathfrak{A} \cup \{T \mapsto X\} \models \gamma$ if and only if $\mathfrak{A} \models_X \eta$. We claim that $\varphi \Rightarrow \eta$ and $\psi \Rightarrow \neg\eta$. Let $\mathfrak{A} \models_X \varphi$. Since $\varphi \Rightarrow \varphi^{\downarrow}$ and $\varphi^{\downarrow}$ is 1-closed, $\mathfrak{A} \models_{\{\bar{x}\}} \varphi^{\downarrow}$ for $\bar{x} \in X$. Using $\{\bar{x}\}$ for T we can construct a model of φ°, which implies $\mathfrak{A} \cup \{T \mapsto \{\bar{x}\}\} \models \gamma$. Thus $\mathfrak{A} \models_{\{\bar{x}\}} \eta$. Since η is a first-order formula, the satisfaction for all subteams of cardinality 1 implies $\mathfrak{A} \models_X \eta$ (lemma 9).

Now assume $\mathfrak{A} \models_X \psi$. Using 1-closedness again we get $\mathfrak{A} \models_{\{\bar{x}\}} \psi^{\downarrow}$ for every $\bar{x} \in X$. We get a model of $\varphi^{\circ} \wedge \exists \bar{y} T \bar{y}$ interpreting $T \mapsto \{\bar{x}\}$ (of course $\exists \bar{y} T \bar{y}$ gets satisfied by such an interpretation). Thus $\mathfrak{A} \cup \{\bar{x}\} \models \neg\gamma$ and $\mathfrak{A} \models_{\{\bar{x}\}} \neg\eta$ (the negation is classical since η is a first-order formula). This implies $\mathfrak{A} \models_X \neg\eta$. □

Corollary 21. *Theorem 16 can be applied to contradictory sentences of such a logic, contradictory formulas of dependence logic ([13], theorem 3.1) and contradictory formulas of independence logic of the form $\exists \bar{x} \varphi$ where the variables used in the independence atoms are free variables of φ because contradictoriness implies strong contradictoriness for these formulas according to lemma 17.*

The restriction to models of cardinality ≥ 2 might appear unnatural, however, it was necessary for our theorem since for cardinalities ≤ 1 tertium non datur holds:

- Let $\mathfrak{A}$ be a structure of cardinality 0, φ a $\mathcal{D}$-formula in prenex normal form and $X \neq \emptyset$ a team. Then φ is actually a sentence and $\mathcal{A} \models_X^{\mathcal{D}} \varphi$ iff the first quantifier is $\forall$ or φ is quantifier-free and holds regarded as a boolean expression where $=()$ is regarded as $\top$ and $\neg =()$ is regarded as $\bot$.

- Let $\mathcal{A}$ be a structure of cardinality 1, φ a $\mathcal{D}$-formula in prenex normal form and $X \neq \emptyset$ a team. Then all quantifiers can be removed and the variables replaced with a constant symbol c if $X = \{\emptyset\}$ or with the free variable c assigned by X

otherwise. $\mathcal{A} \vDash^{\mathcal{D}}_X \varphi$ iff φ holds regarded as a boolean expression where $=()$ and $=(c)$ are regarded as $\top$ and $\neg =()$ and $\neg =(c)$ are regarded as $\bot$.

6 A model-checking game for inclusion logic

Galliani and Hella have proved that for every sentence of inclusion logic there is an equivalent sentence in *posGFP*, the fragment of greatest fixed point logic where gfp-operators occur only positively.[8] Thus model-checking games for inclusion logic are given by safety games. Using a very similar approach we can also directly construct such games.

Intuitively the game can be described as follows: We play a usual model-checking game as for first-order logic. In positions corresponding to occurences of subformulas $\bar{x} \subset \bar{y}$ the proponent tries to give an assignment $\bar{b}$ assigning the variables $\bar{y}$ in agreement with the previous assignment of the variables $\bar{x}$. Now the opponent tries to prove that this assignment $\bar{b}$ cannot be an element of any team considered at the position $\bar{x} \subset \bar{y}$. Therefore she moves back to an occurence of a subformula containing the mentioned occurence of the form $\bar{x} \subset \bar{y}$ as a subformula and tries to move to a contradiction given that part of the information in $\bar{b}$ which is already known at this subformula.

Definition 9. Given a structure $\mathfrak{A}$, a formula φ of inclusion logic such that no variable occurs both freely and bound in any subformula and a team $X \subset A^{fr(\varphi)}$. We define the model-checking game $MC(\mathfrak{A}, X, \varphi)$ as follows:

- There is a special initial position $\sharp$
- Every pair $(\psi, \bar{a})$ consisting of an occurence of a subformula ψ of φ which does not occur as a proper subformula of a negated formula and an assignment $\bar{a} \in A^{var(\psi)}$ represents a position of $MC(\mathfrak{A}, X, \varphi)$.
- For each occurence of a subformula ψ of the form $\bar{x} \subset \bar{y}$ and each assignment $\bar{a} \in A^{var(\psi)}$ we introduce a second position $(\psi, \bar{a})'$.

Now for every state we will give all possible moves (the first three rules are special, the rest is analogous to first-order logic):

- $\sharp$: The opponent can move to any position $(\varphi, \bar{a})$ where $\bar{a} \in X$.
- $(\psi, \bar{a})$ where ψ is of the form $\bar{x} \subset \bar{y}$: The proponent can move to any position $(\psi, \bar{b})'$ such that $\bar{b} \upharpoonright var(\varphi) \in X$ and $[\![\bar{x}]\!]^{\bar{a}} = [\![\bar{y}]\!]^{\bar{b}}$.
- $(\psi, \bar{a})'$: The opponent can move to any position $(\chi, \bar{a} \upharpoonright var(\chi))$ where ψ is contained in χ.
- $(\psi \vee \chi, \bar{a})$: The proponent can move to $(\psi, \bar{a})$ or $(\chi, \bar{a})$.
- $(\psi \wedge \chi, \bar{a})$: The opponent can move to $(\psi, \bar{a})$ or $(\chi, \bar{a})$.
- $(\exists x\psi, \bar{a})$: The proponent can move to any $(\psi, \bar{a} \cup \{x \mapsto b\})$ where $b \in A$.

- $(\forall x\psi, \bar{a})$: The opponent can move to any $(\psi, \bar{a} \cup \{x \mapsto b\})$ where $b \in A$.

Now we will define the winning-conditions:

- In positions $(\psi, \bar{a})$ where ψ is of the form $R\bar{x}$ (for any relation symbol R including equality) the proponent wins if and only if $[\![\bar{x}]\!]^{\bar{a}} \in R^{\mathfrak{A}}$.
- For positions $(\psi, \bar{a})$ where ψ is of the form $\neg R\bar{x}$ we use the opposite condition.
- At positions $(\psi, \bar{a})$ where ψ is of the form $\neg \bar{x} \subset \bar{y}$ the opponent wins.
- In every infinite match the proponent wins (safety condition).

Theorem 22. *$\mathfrak{A} \models_X \varphi$ if any only if the proponent has a winning strategy in $MC(\mathfrak{A}, X, \varphi)$.*

Proof. Assume that $\mathfrak{A} \models_X \varphi$. Let (X_ψ) be a witnessing family of teams for φ and X. We claim that a strategy for the proponent obeying the following rules is a winning strategy:

- From position $(\psi_0 \vee \psi_1, \bar{a})$ $(\bar{a} \in X_{\psi_0 \vee \psi_1})$ move to $(\psi_i, \bar{a})$ such that $\bar{a} \in X_{\psi_i}$.
- From position $(\exists x\psi, \bar{a})$ $(\bar{a} \in X_{\exists x\psi})$ move to $(\psi, \bar{a} \cup \{x \mapsto b\})$ where $\bar{a} \cup \{x \mapsto b\} \in X_\psi$.
- From position $(\bar{x} \subset \bar{y}, \bar{a})$ $(\bar{a} \in X_{\bar{x} \subset \bar{y}})$ move to $(\bar{x} \subset \bar{y}, \bar{b})'$ where $\bar{b} \in X_{\bar{x} \subset \bar{y}}$ such that $[\![\bar{x}]\!]^{\bar{a}} = [\![\bar{y}]\!]^{\bar{b}}$.

We notice that for every subformula η of a subformula ψ $X_\psi \subset X_\eta \upharpoonright var(\psi)$ holds. Using this fact we prove that every move of the opponent from position $(\bar{x} \subset \bar{y}, \bar{b})'$ (in a match where the proponent used her strategy) will result in a position $(\psi, \bar{b} \upharpoonright var(\psi))$ where $\bar{b} \upharpoonright var(\psi) \in X_\psi$. We conclude that every such strategy is actually a winning strategy.

Now assume that the proponent has a winning strategy in $MC(\mathfrak{A}, X, \varphi)$. For every subformula ψ of φ which is not a proper subformula of a negated formula let W_ψ denote the set of all assignments $\bar{a} \in A^{var(\psi)}$ such that $\bar{a} \upharpoonright var(\varphi) \in X$ and such that the proponent has a winnig strategy from the position $(\chi, \bar{a} \upharpoonright var(\chi))$ for every subformula χ containing ψ as a subformula. We notice that $W_\varphi = X$, since the proponent has a winning strategy from every position $(\varphi, \bar{a})$ for $\bar{a} \in X$. Actually the family (W_ψ) is a witnessing family of teams for X, φ, $\mathfrak{A} \models_{W_\psi} \psi$ for all occurences of subformulas ψ of φ:

- The case where ψ is of the form $R\bar{x}$ or $\neg R\bar{x}$ is clear.
- ψ is of the form $\bar{x} \subset \bar{y}$: Given any $\bar{a} \in W_\psi$: The proponent has a winning strategy from $(\psi, \bar{a})$. According to the definition of her moves and the definition of W_ψ she can choose $\bar{b} \in W_\psi$ such that $[\![\bar{x}]\!]^{\bar{a}} = [\![\bar{y}]\!]^{\bar{b}}$. This implies $\mathfrak{A} \models_{W_\psi} \psi$.
- Negated inclusion atoms are trivial.
- $\psi = \chi \vee \eta$: Notice that $W_\psi = W_\chi \cup W_\eta$.
- $\psi = \chi \wedge \eta$: Notice that $W_\psi = W_\chi = W_\eta$.

- $\psi = \exists x \chi$: Notice that $W_\psi = W_\chi \upharpoonright var(\psi)$.
- $\psi = \forall x \chi$: Notice that $W_\psi = W_\chi(A/x)$.

□

A very similar model-checking game could be constructed for logics with team semantics using only connectives defined by first-order formulas of the form $\forall \bar{x}(R\bar{x} \to \theta(\bar{x}))$ where R (representing the team) occurs only positively in θ (cf. [8], p. 16). Since teams satisfying such a predicate are closed under unions we can apply the same arguments involving maximal teams. When arriving at the position corresponding to such an atomic formula we have to embed the (first-order) model-checking game for θ into the game and when this game reaches a position corresponding to a subformula $R\bar{y}$ the opponent is allowed to move "upwards in the syntax tree" analogously to the game defined before.

The following example illustrates the expressive power of inclusion logic.

Example 3. *There exists a sentence φ in inclusion logic such that $\mathfrak{A} = (U, E) \models \varphi$ if and only if E is not a well-founded relation.*

Proof. Choose $\varphi = \exists x \exists y(Eyx \wedge y \subset x)$. In the model-checking game the proponent has to move along the edges in E (otherwise she looses). She has a winning strategy if and only if there is such an infinite path. □

This property is not definable in first-order logic. However, sentences of inclusion logic do not capture the expressive power of Σ_1^1: On finite structures Σ_1^1-definable properties are precisely the NP-properties. But lfp does not even capture all of P.

7 A Löwenheim-Skolem theorem for inclusion logic

For lfp over a countable signature a certain downward Löwenheim-Skolem theorem holds: Every structure over a countable signature has a countable *lfp-substructure* ([5]). Especially every satisfiable lfp-formula has a countable model and every non-valid lfp-formula has a countable non-model since lfp is closed under classical negation. Using this result we get a Löwenheim-Skolem theorem for *sentences* of inclusion logic. However, it is possible to give a similar statement specifically related to team semantics, which we will state now.

Theorem 23. *Given a structure $\mathfrak{A}$ over a countable signature. Then there exists a countable elementary substructure $\mathfrak{A}'$ such that for every $\varphi \in \mathcal{INC}$ and maximal teams $X \subset A^v$ such that $\mathfrak{A} \models_X \varphi$ we have that $A'^v \cap X$ is the maximal team on $\mathfrak{A}'$ satisfying φ.*

Proof. First we choose a fixed countable, ordered set of variables. In this section $\mathcal{INC}$ shall denote only those formulas of inclusion logic only using these variables. Now we choose any well-order on A using the well-ordering theorem. For every formula φ this induces a unique well-order on the positions in the unfolding of $MC(\mathfrak{A}, X, \varphi)$ to a tree, called $MC(\mathfrak{A}, X, \varphi)'$, satisfying the following conditions:

- The positions are ordered by the number of turns till they are reached.
- If a position requires choosing an element of A, the positions following-up are ordered correspondingly by the well-order on A.
- Positions corresponding to the left-hand side of an occurence of a subformula $\varphi \vee \psi$ or $\varphi \wedge \psi$ are prior to positions corresponding to the right-hand side.

Using this well-order we get a partial order on strategies by component-wise comparisons. Every set of strategies given by restricting the set of positions has a minimum with respect to this order, especially there is a minimal winning strategy for the winning player.

Now we construct the desired substructure. Let $e\colon \mathbb{N} \to \mathcal{INC} \times \mathbb{N} \times \mathbb{N} \cup \mathcal{INC} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be a (bijective) enumeration such that $e(n)_2 \leq n$ and $e(n)_3 < e(m)_3$ for all $n < m \in \mathbb{N}$. We define $A_0 := \emptyset$ and inductively:

- The game $M_{\varphi,n}$ is given by restricting $MC(\mathfrak{A}, X, \varphi)'$ where X is the maximal team on $\mathfrak{A}$ satisfying φ to the maximal game tree contained in it satisfying the following conditions:
  - Every position is in the winning region of the proponent.
  - At the positions of the proponent she can only move to the minimal succeeding position which is still in her winning region.
  - At the positions of the opponent involving a choice of elements of A she can only choose elements of A_n and elements of A which have been chosen before in the game.
- The game $N_{\varphi,n,s}$ is given by restricting $MC(\mathfrak{A}, A^{fr(\varphi)}, \varphi)'$ where $s \in A_n^{fr(\varphi)} \setminus X$ and X is the maximal team on $\mathfrak{A}$ satisfying φ to the maximal game tree contained in it satisfying the following conditions:
  - Every position is in the winning region of the opponent.
  - At the start position the opponent has to select the assignment s.
  - At the positions of the proponent involving a choice of elements of A she can only choose elements of A_n and elements of A which have been chosen before in the game.
  - At the positions of the opponent she can only move to the minimal succeeding position which is still in her winning region.
- If $e(n)$ is a 3-tuple, then $G_n = M_{e(n)_1, e(n)_2}$.
- If $e(n)$ is a 4-tuple, then $G_n := N_{e(n)_1, e(n)_2, s}$ where s is the $e(n)_4$-th assignment in $A_{e(n)_2}^{fr(\varphi)} \setminus X$ using the natural well-order on this finite product of well-ordered sets where X is the maximal team on $\mathfrak{A}$ satisfying φ, assuming that $e(n)_4$ does not reach or exceed the number of elements of this set. Otherwise set $G_n := \emptyset$.

- If $G_n = \emptyset$, then set $A_{n+1} = A_n$. Otherwise let A_{n+1} be the union of A_n and the set of all elements of A' which have been used in a move of a player in G_n somewhere in the first $e(n)_3$ positions.

Notice that the games $M_{\varphi,n}$ and $N_{\varphi,n,s}$ are finitely branching guaranteeing that the well-order on their positions is isomorphic to an ordinal $\leq \omega$, thus all the positions get considered at some point to extend a set A_m, which is implied by the finiteness of every set A_n. This finiteness also implies that $A' := \bigcup_{i\in\mathbb{N}} A_i$ is countable.

We prove that for the maximal team X on $\mathfrak{A}$ satisfying φ the team $X' := X \cap A'^{fr(\varphi)}$ is the maximal team on $\mathfrak{A}'$ satisfying φ for each formula φ. First we show that the minimal winning strategy of the model-checking game for X can be restricted to a winning strategy on X'. Assume that there exists a position in the winning region of the proponent in $MC(\mathfrak{A}, X, \varphi)$ which is not in the winning region of the proponent in $MC(\mathfrak{A}', X', \varphi)$. With respect to the well-order on the positions of $MC(\mathfrak{A}, X, \varphi)$ given before there exists a minimal position g amongst these positions. At g a finite set of elements $S \subset A$ has already been considered and the proponent has to choose a certain assignment of variables (either in the context of existential quantification or in the context of an inclusion atom). Th minimal possible choice $\bar{a} \subset A$ remaining in the winning region gets selected by the minimal winning strategy. However, at some step n in the construction of A' above all the variables in S have already been selected. Thus at some position (φ, n, s) in the construction above the position g is reached, thus the values in $\bar{a}$ are contained in A' (since it is the minimal choice remaining in the winning region of the proponent). A contradiction, thus the proponent wins $MC(\mathfrak{A}', X', \varphi)$.

It remains to be proved that X' is maximal. Consider any team $Y \subset A'^{fr(\varphi)}$ such that $Y \not\subset X'$, choose $s \in Y \setminus X'$. We notice that $s \notin X$. Thus the opponent has a winning-strategy in $MC(\mathfrak{A}, Y, \varphi)$ where he starts by choosing the assignment s. The minimal one of these strategies can be restricted to a winning strategy in $MC(\mathfrak{A}', Y, \varphi)$.

By applying this result to first-order formulas with free variables we also establish that $\mathfrak{A}'$ is an elementary substructure of $\mathfrak{A}$. □

Example 4. *We remark that for general teams X on $\mathfrak{A}$ the statements $\mathfrak{A} \models_X \varphi$ and $\mathfrak{A}' \models_{X\cap A'^{fr(\varphi)}} \varphi$ do not have to imply each other and this cannot be changed by choosing another construction of A'. Consider the real line $\mathfrak{A} = (\mathbb{R}, <, 0)$, without loss of generality any countable elementary substructure of $\mathfrak{A}$ can be regarded as $\mathfrak{A}' = (\mathbb{Q}, <, 0)$. Let $\varphi := 0 < x \wedge \exists y x < y \wedge y \subset x$. The maximal team on $\mathfrak{A}$ satisfying φ is $\mathbb{R}_+$ and the maximal team on $\mathfrak{A}'$ satisfying φ is $\mathbb{Q}_+ = \mathbb{R}_+ \cap \mathbb{Q}$. However, let $E := \mathbb{R}_+ \setminus \mathbb{Q} \cup \{1\}$ and $F := \mathbb{R} \setminus \mathbb{Q}_{\leq 0}$. Then $\mathfrak{A} \models_E \varphi$ and $\mathfrak{A} \not\models_F \varphi$, but $\mathfrak{A}' \not\models_{E\cap\mathbb{Q}} \varphi$ and $\mathfrak{A}' \models_{F\cap\mathbb{Q}} \varphi$.*

8 Extensions of inclusion logic

In this section we will consider some extensions of inclusion logic using connectives which have been studied before to extend dependence logic.

First we will consider adding classical, Boolean negation to inclusion logic. We get the following theorem:

Theorem 24. *The logic gained from inclusion logic by adding classical, Boolean negation is as expressive as team logic.*

Proof. There is a formula to express exclusion $(\bar{x}|\bar{y})$, namely

$$\sim((\exists \bar{z}(\bar{z} \subset \bar{x} \wedge \bar{z} \subset \bar{y})) \wedge \sim\bot)$$

where $\bar{z}$ is a tuple as long as $\bar{x}$ and $\bar{y}$ consisting of new, distinct variables. To sets are disjoint if and only if there is no non-empty common subset. The case of the empty team is treated separately using $\sim\bot$. Since exclusion logic is equivalent to dependence logic we get full team logic. □

Let us consider intuitionistic and maximal implication:

Theorem 25. *The logic gained from inclusion logic by adding intuitionistic implication is as expressive as second-order logic with respect to sentences.*

Proof. Exclusion $(\bar{x}|\bar{y})$ can be expressed as an *intuitionistic negation* of inclusion:

$$\bar{x} \subset \bar{y} \to \bot.$$

This formula expresses that every subteam satisfying $\bar{x} \subset \bar{y}$ of a given team is in fact empty, which is obviously equivalent to exclusion. Thus this logic is at least as strong as intuitionistic dependence logic, which is equivalent to second-order logic on sentences. We notice that we could have used maximal implication as well and that intuitionistic negation suffices to get the power of independence logic. □

Now we will consider linear implication:

Theorem 26. *The model-checking problem on finite structures for the logic gained from inclusion logic by adding linear implication is $Co - NP$ hard.*

Proof. We will construct a formula such that checking whether certain finite structures are a model of that formula gives an answer to the problem whether a certain propositional formula in disjunctive normal form is a tautology. We encode instances of that problem as structures over a signature $\{V, P, N, C\}$ where V, P are unary relation symbols and N, C are binary relation symbols. Given a formula φ in disjunctive normal form we construct a structure $\mathfrak{A}$ on the universe A with the following properties:

- $V^{\mathfrak{A}}$ is the set of all variables in φ and their negations.
- $P^{\mathfrak{A}} \subset V^{\mathfrak{A}}$ is the set of all variables in φ.
- $N^{\mathfrak{A}}$ associates each variable in φ to its negation and vice-versa.
- $A \setminus V^{\mathfrak{A}}$ is the set of all monomials in φ.
- $C^{\mathfrak{A}} \subset (A \setminus V^{\mathfrak{A}}) \times V^{\mathfrak{A}}$ associates to every monomial in φ the variables and negated variables it consists of.

Now we will give the desired formula:

$$\exists x \neg V x \wedge ((\forall y(\neg V y \vee y \subset x \vee \exists z(N y z \wedge z \subset x))) \multimap \exists m(\neg V m \vee \forall y(\neg C m y \vee y \subset x)))$$

The left hand side of the linear implication $(\forall y(\neg V y \vee y \subset x \vee \exists z(N y z \wedge z \subset x)))$ expresses that we want to consider teams X such that for each variable x in φ we have $x \in X$ or $\neg x \in X$. The right hand side expresses that the DNF formula gets satisfied when interpreting those variables/negated variables as true which are contained in X. The subformula $\neg V x$ guarantees that we are actually considering all those teams, the existential quantifier only introduces meaningless monomials into the team. □

Corollary 27. *If $P \neq NP$ holds, then this logic cannot be translated neither to Σ_1^1 nor to fixed-point logic.*

It remains an interesting open question whether there are some natural connectives extending inclusion logic such that we get the expressive power of full fixed-point logic.

9 Effective characterisations

9.1 Axiomatising consequences

For sufficiently strong logics with team-semantics there is no hope to get a completeness theorem: If we can recursively enumerate all semantic consequences in a logic, then we can also enumerate all tautologies by starting from a tautology. We will now see that the set of all closed tautologies in inclusion logic (and thus the sets of all tautologies in inclusion, dependence and independence logic) is not recursively enumerable, in fact we will prove that these sets are not even arithmetical. The set of all tautologies in a logic will be called the *decision problem* of the logic.

Theorem 28. *For every first-order sentence φ in the language of arithmetics (using $+, \cdot, <$) we can compute a sentence ψ in inclusion logic which is a tautology if and only if φ is not a theorem of true arithmetics (i. e. valid for the natural numbers).*

Proof. Let π be a first-order sentence axiomatising the fragment PA^- of Peano arithmetics (for a definition see [9], it does not include any induction principles). Every model of π contains the natural numbers as an initial segment. You can easily prove (or assume it as an axiom) that every non-zero element of a model of PA^- has a predecessor. Since the natural numbers are the only infinite well-order where 0 is the only element without a predecessor, every well-ordered model of PA^- is isomorphic to the natural numbers. Let χ be a sentence of inclusion logic expressing that $<$ is not a well-order (see example 3). Then the models of $\neg\pi \wedge \chi$ are precisely those structures which are not isomorphic to the natural numbers. Choose $\psi = (\neg\pi \wedge \chi) \vee \varphi$, then φ is a theorem of true arithmetics if and only if ψ is a tautology. □

By Tarski's undefinability theorem set of all tautologies of inclusion logic is not arithmetical. In the next section we will refine these results.

However, there is hope if we want to enumerate only *certain* consequences of formulas in logics with team semantics translatable to Σ_1^1. The simplest problem is the *inconsistency problem*: We want to decide whether the falsum is implied by a formula. There we get a general positive result.

Lemma 29. *Inconsistency in logics effectively translatable to Σ_1^1 is semi-decidable.*

Lemma 30. *If a logic L is effectively translatable to Σ_1^1 and contains first-order logic (providing an effective embedding), then its inconsistency problem is recursively isomorphic to the inconsistency problem of first-order logic.*

Proof. It is well known that the inconsistency problem for first-order logic is a complete, semi-decidable problem. The effective translation to Σ_1^1 induces an m-reduction of the inconsistency problem to the inconsistency problem of first-order logic. The effective embedding of first-order logic induces an m-reduction of the inconsistency problem of first-order logic to the inconsistency problem of L. Thus the inconsistency problem of L is m-complete, thus it is 1-complete (see [19], p. 87), thus it is recursively isomorphic to the halting problem and the inconsistency problem of first-order logic according to Myhill's isomorphism theorem (see [19], p. 85). □

Corollary 31. *The first-order consequences φ of every recursively enumerable set Φ of formulas in such a logic are recursively enumerable.*

Proof. φ is a consequence of Φ if and only if $\Phi \cup \{\varphi\}$ is inconsistent. By the compactness theorem of first-order logic this is the case if and only if there is a finite inconsistent subset of $\Phi \cup \{\varphi\}$. The inconsistency of a finite set is semi-decidable by lemma 29. Since the finite subsets of sets of the form $\Phi \cup \{\varphi\}$ are recursively enumerable, the corollary holds. □

In [14] an explicit schema to axiomatise the first-order consequences of dependence logic has been constructed (which in principle also gives a schema for the first-order consequences of more general logics with team semantics by adding ruls guaranteeing that $\varphi^\downarrow$ is derivable from φ, since φ and $\varphi^\downarrow$ have the same first-order consequences.

9.2 The Levy hierarchy

We can also give some statements about the degrees of unsolvability of the decision problem of logics with team semantics. Väänänen has proved that the decision problem of dependence logic is Π_2-complete in the Levy hierarchy ([20], theorem 7.5). We will briefly present how the Levy hierarchy can be used to establish recursion theoretic properties of languages, giving some details which are omitted in [20], but building upon some well-known theorems.

Definition 10. A formula in the language of set theory is called a Δ_0-, Σ_0- or Π_0-formula (in the Levy hierarchy) if all its quantifications are bounded (i. e. they are of the form $\exists x(x \in y \wedge \varphi)$, which we will denote by $\exists x \in y\varphi$, or $\forall x(x \in y \rightarrow \varphi)$, which we will denote by $\forall x \in y\varphi$). Inductively a Σ_{n+1}-formula ($\Pi_{n+1)}$) is any formula of the format $\exists x\varphi$ where φ is a Π_n-formula ($\forall x\varphi$ where φ is a Σ_n-formula, respectively) for natural numbers n. A formula which is constructed from $\Sigma_0, \dots \Sigma_n$ formulas by Boolean combinations and bounded quantification will be called a B_n-formula.

We will subsume some closure properties:

Lemma 32. *In ZF equivalences of the following types can be proved:*

- *A Π_n-formula is equivalent to the negation of a Σ_n-formula.*
- *If φ is a Σ_n-formula, then $\exists x\varphi$ is equivalent to a Σ_n-formula.*
- *If φ is a Π_n-formula, then $\forall x\varphi$ is equivalent to a Π_n-formula.*
- *If φ, ψ are Σ_n-formulas (Π_n-formula), then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are equivalent to Σ_n-formulas (Π_n-formulas).*
- *If φ is a Σ_n-formula (Π_n-formula), then $\exists x \in y\varphi$ and $\forall x \in y\varphi$ are equivalent to Σ_n-formulas (Π_n-formulas), which is proved using the axiom of replacement.*

Furthermore we are interested in arithmetic statements:

Lemma 33. *Let $n > 0$. In ZF we can prove for every Σ_n-formula (Π_n-formula) φ and every arithmetic formula ψ where X occurs only positively that $\psi'(\varphi/X)$ is equivalent to a Σ_n-formula (Π_n-formula) (ψ' shall be a translation of ψ into the language of set theory defined by introducing a variable $\mathbb{N}$, the natural numbers, and replacing $\exists x, \forall x$ by $\exists x \in \mathbb{N}, \forall x \in \mathbb{N}$ and $+, \cdot$ by their definitions).*

Proof. We consider the Σ_n-case and prove by induction that for every such formula ψ (without loss of generality in negation-normal form) there exists a Σ_n-formula $\rho(\psi)$ such that $\psi'(\varphi/X)$ is equivalent to $\exists\mathbb{N}(\varphi_{\mathbb{N}} \wedge \rho(\psi))$, which is also equivalent to a Σ_n-formula (where $\varphi_{\mathbb{N}}$ is a Δ_0-definition of the set of all natural numbers).

- Set $\rho(R) := \varphi$.
- For $\psi = (x = y + z)$ ($\psi = (x \neq y + z)$, $\psi = (x = y \cdot z)$, $\psi = (x \neq y \cdot z)$) use Σ_1- and Π_1-definitions of additions and multiplication to get a Σ_1-formula $\rho(\psi)$ ([10], p. 505).
- If $\psi = \psi_0 \vee \psi_1$ ($\psi = \psi_0 \wedge \psi_1$) choose $\rho(\psi)$ as a Σ_n-formula equivalent to $\rho(\psi_0) \vee \rho(\psi_1)$ ($\rho(\psi_0) \wedge \rho(\psi_1)$).
- If $\psi = \exists x\psi_0'$ ($\psi = \forall x\psi_0'$) choose $\rho(\psi)$ as a Σ_n-formula equivalent to $\exists x \in \mathbb{N}\rho(\psi_0)$ ($\forall x \in \mathbb{N}\rho(\psi_0)$).

□

The following result by Levy will allow us to define classes of Σ_n-/Π_n-definable languages.

Theorem 34. *For every $n > 0$ there is a Σ_n-definition (Π_n-definition) of truth of Σ_n-formulas (Π_n-formulas), i. e. for every $m \geq 0$ there exists a Σ_n-formula $T_{\Sigma_n,m}(p, x_, \ldots, x_m)$ such that for every Σ_n-formula $\psi(x_1, \ldots, x_m)$ the statement*

$$\forall x_1 \ldots x_m \psi(x_1, \ldots, x_m) \leftrightarrow T_{\Sigma_n,m}(\#\psi, x_1, \ldots, x_m)$$

is provable in ZF where $\#\psi$ is the Gödel number of ψ ([16], p. 26, thm. 20). Since B_n-formula can recursively be transformed into an equivalent Σ_{n+1}-formula, there are also definitions of truth of B_n-formulas, called $T_{B_n,m}$. These satisfy:

- *For all $m \leq m_0, m_1$ and $n \leq n_0, n_1$ the following statement is provable in ZF:*

$$\begin{aligned} &\forall p x_1 \ldots x_{\max\{m_0,m_1\}} \\ &(\text{“}p \text{ is a Gödel number of a } B_n\text{-formula using the first } m \text{ variables”} \\ &\rightarrow (T_{B_{n_0},m_0}(p, x_1, \ldots, x_{m_0}) \leftrightarrow T_{B_{n_1},m_1}(p, x_1, \ldots, x_{m_1}) \end{aligned}$$

- *For all n, m the following statement is provable in ZF:*

$$\begin{aligned} &\forall p q x_1 \ldots x_m \\ &((\text{“}p, q \text{ are Gödel numbers of } B_n\text{-formulas using the first } m \text{ variables”} \\ &\rightarrow ((T_{B_n,m}(p, x_1, \ldots, x_m) \vee T_{B_n,m}(q, x_1, \ldots, x_m)) \\ &\leftrightarrow T_{B_n,m}(\text{“Gödel number of the disjunction of the formulas represented by } p, q\text{”}, \\ &x_1, \ldots x_m)))) \end{aligned}$$

- *Analogous conditions for conjunction, negation and (bounded and unbounded) quantification.*

Using these definitions we get direct notions of Σ_n-/Π_n-definability without having to assume the existence of a model of ZF and without refering to the notion of provability, thus the following definitions do not operate in a meta-theory, but directly defines sets of languages Σ_n, Π_n.

Definition 11. Let $n \geq 0$. $\Sigma_n \subset \mathcal{P}(\mathbb{N})$ is the set of all Σ_n-definable subsets of $\mathbb{N}$, more precisely: A subset $A \subset \mathbb{N}$ is an element of Σ_n if and only if there exists a valid Gödel number p such that $\forall x \in \mathbb{N}(x \in A \leftrightarrow T_{\Sigma_n,1}(p, x))$. Π_n is defined analogously. $\Delta_n := \Sigma_n \cap \Pi_n$.

The following lemma assures us that we can transfer some meta-theoretical statements involving provability into statements inside ZF:

Lemma 35. *For every $n \geq 0$ the following statement is provable in ZF: Every B_n-axiom of ZF is true.*

Proof. For any finite set of axioms the proof is immediate by inserting their Gödel numbers into the truth predicate for B_n. It remains to consider the B_n-instances of the axiom schema of replacement (which implies separation): The truth of every instance of this schema is expressed by

$$\forall p \forall m(\text{“}p\text{ is a Gödel-number of a }B_{n-1}\text{-formula with }m\text{ free variables”} \\ \to T_{B_n,0}(\#(\forall x_1 \dots x_m \forall y \exists z(\\ \text{“if }p\text{ represents a functional predicate, }z\text{ is the image of }y\text{ under this predicate”}))).$$

Using theorem 34 the quantification $\forall x_1 \dots x_m$ inside the construction of a number of a formula can be transformed into a quantification $\forall x$ over cartesion products outside of the $T_{B_n,0}$ and the application of $T_{B_n,0}$ can be put inside an instance of the axiom schema of replacement, replaced by a formula $\forall q \chi(p, q) \wedge T_{B_n,3}(q)$ where $\chi(p, q)$ states that the formula represented by q is constructed by that one represented by p by replacing every occurence of x_i by some expression to access the i-th component of x. Since this equivalent formula is an instance of the axiom schema of replacement, it is provable. □

This lemma establishes the following meta-mathematical argument: Whenever we have a meta-mathematical proof that a certain class of Σ_n-statements is provable by proofs only involving Σ_m-statements for some $m \geq n$ only depending on n, then we get a proof in ZF about truth of an analogously defined class of Σ_n-statements, since we can prove in ZF that “provability within Σ_n” implies “truth within Σ_n”. By considering how any Σ_n or Π_n formula can be turned into both equivalent Σ_{n+1}- and Π_{n+1}-formulas this principle can be used to prove that we have actually defined a hierarchy of languages:

Lemma 36. *For every $n \geq 0$ the statement $\Sigma_n, \Pi_n \subset \Delta_{n+1}$ is a theorem of ZF.*

We will now apply the same principle to derive some stability of this hierarchy under recursive operations of Σ_n-sets from the proof-theoretic statement of lemma 33.

Lemma 37. *$\Sigma_n, \Pi_n, \Delta_n$ are closed under totally recursive preimages.*

Proof. We prove the lemma for Σ_n, then the cases of Π_n and Δ_n follow automatically. Given any totally recursive function $f \colon \mathbb{N} \to \mathbb{N}$. There is an arithmetic formula $\varphi(x, X)$ (where X is a unary relation symbol) expressing that $f(x) \in X$. By lemma 33 □

Now we want to prove that this hierarchy of languages does not collapse. We get this result by combining three theorems: The definability of truth in Σ_n (theorem 34), Tarski's theorem of the undefinability of truth and the Levy-Montague reflection principle. We will now state the latter one:

Theorem 38. *For every formula $\varphi(x_1, \dots, x_m)$ in ZF the following sentence is provable in ZF:*

$$\exists U(U \textit{ transitive } \wedge \forall x_1 \dots x_m \varphi \leftrightarrow \varphi^U)$$

where φ^U denotes the relativisation of φ to U ($\exists x, \forall x$ get replaced by $\exists x \in U, \forall x \in U$) ([15], p. 228). In fact this schema is equivalent to the schema of replacement plus the axiom of infinity in the presence of the remaining axioms of ZF.

For $m = 0$ the reflection principle implies that ZF proves the consistency of any finite fragment of ZF, but we can get an even stronger statement related to the Levy hierarchy:

Theorem 39. *For every $n \geq 0$ we can prove the consistency of the set of all true $\Sigma_n, \Pi_n, \Delta_n$-sentences.*

Proof. It suffices to consider Σ_n-sentences. Let $n > 0$. There is a Σ_n-formula $\varphi(x)$ equivalent to the statement $\exists \mathbb{N}(\varphi_{\mathbb{N}} \wedge$ "x is the Gödel number of a Σ_n-sentence" $\wedge\, T_{\Sigma_n,0}(x))$. Now the Levy-Montague reflection principle gives a model of the set of all true Σ_n-sentences by observing that relativising $T_{\Sigma_n,0}$ to a set gives a definition of model-theoretic satisfaction. □

Theorem 40. *The hierarchy $\Delta_n, \Sigma_n, \Pi_n$ does not collapse, i. e. for all $n \geq 0$ we can prove that $\Sigma_n \neq \Pi_n$.*

Proof. Assume that $\Sigma_n = \Pi_n = \Delta_n$ and $n > 0$. By theorem 34 there is a Σ_n-definition of the set of all true Σ_n-sentences. Since $\Sigma_n = \Pi_n$, there is also a Π_n-definition of this set inducing a Σ_n-definition of falsity of Σ_n-sentences implying that Σ_n-sentences can recursively transformed into a Σ_n-sentence equivalent to their negation (by inserting the Gödel number into the definition of falsity). Thus all Boolean combinations of Σ_n-sentences can recursively transformed into equivalent Σ_n-sentences. Applying lemma 33 we see that we can also give a Σ_n-definition of validity of Σ_n-sentences with natural number variables and recursively perform Boolean combinations of such formulas. We can now regard these possibilities of combinations of Σ_n-formulas as a logic to describe natural numbers. Again by lemma 33 this logic provides enough arithmetics to prove the diagonal lemma. Thus by Tarski's undefinability theorem it cannot be both consistent and providing a definition of truth. But the set of all true Σ_n-sentences is consistent by theorem 39, a contradiction. Notice that we did *not* assume consistency of ZF as an additional axiom to prove this theorem, the proof works in ZF for every n. □

We have now established a hierarchy of languages very similar to the arithmetic hierarchy (but containing much stronger languages), however, there is the substantial difference that all statements of the form "Let $n \geq 0 \ldots\ \Sigma_n \ldots$" have to be read as a meta-mathematical schema of statements: One statement in the language of set theory for every natural number n. But for any fixed or bounded n we can now freely use these classes without involvement of meta-mathematics.

9.3 Decision problems

Theorem 41. *The decision problem (decide whether a sentence is a tautology) of higher-order logic is in Π_2 in the Levy hierarchy.*

Proof. Suppose φ to be a formula of higher-order logic. φ is a tautology if and only if

$$\varphi' := \forall \mathfrak{A} v(\mathfrak{A}) \wedge \forall V(u(A, V) \to \tilde{\varphi})$$

holds where:

- $v(\mathfrak{A})$ denotes that $\mathfrak{A}$ is defined on the same signature as φ, obviously that is possible in Δ_0 using Δ_0-definitions of operations with tuples.
- $u(A, V) := A \in V \wedge \forall x \in V \exists y \in V \forall z \in yz \subset x \wedge \forall z(z \subset x \to z \in y) \wedge \forall a, b \in V \exists y \in V \forall z \in y \exists g \in a \exists h \in bz = (g, h) \wedge \forall g \in a \forall h \in b(a, b) \in y$, a Π_1-formula exprssing that V contains A and its powerset and cartesian products of A and so on.
- $\tilde{\varphi}$ is constructed from φ by replacing $\forall x$ by $\forall x \in A$, $\forall R$ (where R is a dyadic relation symbol) by $\forall a \in V \forall R \in A(\forall x \in R \exists a, b \in Ax = (a, b))$, Rxy by $(x, y) \in R$ and so on.

φ' is equivalent to a Π_2 formula. There is a Π_2-formula defining the set of all Gödel numbers of true Π_2-formulas. Thus the decision problem is in Π_2 because Π_2-problems are closed under recursive preimages. □

Corollary 42. *The decision problems of dependence logic, logics with team semantics containing dependence logic being effectively translatable to Σ^1_1, existential second-order logic, second-order logic, team logic, and higher-order logic are recursively isomorphic.*

Proof. Because the decision problem of dependence logic is Π_2-complete ([20], theorem 7.5) and because all the given logics can be effectively embedded into higher-order logic, whose decision problem is Π_2 according to theorem 41, these decision problems are m-equivalent. But every m-reduction f between these problems can be translated into a 1-reduction g: Choose $g(n)$ to be the Gödel number of the formula $(\bigwedge_{i<n} (A \vee \neg A)) \wedge \varphi_{f(n)}$ where $\varphi_{f(n)}$ is the formula with the Gödel number $f(n)$ and A is a propositional variable not occuring in any $\varphi_{f(n)}$ (if that is not the case, f can be replaced by another recursive function which does not use A). Now g is injective. Myhill's isomorphism theorem implies that the mentioned decision problems are recursively isomorphic. □

For inclusion logic we can prove that its decision problem is not as hard as the decision problem for dependence logic. Using the game-theoretic semantics we will prove that it is in Π_1 in the Levy hierarchy.

Theorem 43. *The decision problem of inclusion logic is in Π_1 in the Levy hierarchy.*

Proof. We will consider the dual problem of non-validity: Safety games (as a special case of parity games) are always positionally determined. Thus a formula of inclusion logic is non-valid if and only if there exists a structure such that there exists a positional winning strategy for the opponent in the corresponding model-checking game. For a formula φ in inclusion logic we construct a corresponding Σ_1-formula φ' which is true if and only if φ is non-valid. We use the following schema:

$$\varphi' := \exists \mathfrak{A} \exists X \exists V \exists E \exists V_0 \exists S v(\mathfrak{A}) \wedge t(\mathfrak{A}, X) \wedge g(\mathfrak{A}, X, V, E, V_0) \wedge w(V, E, V_0, S)$$

The subformulas v, g, w are constructed as follows:

- $v(\mathfrak{A})$ expresses that $\mathfrak{A}$ is a first-order structure with the same signature as φ.

- $t(\mathfrak{A}, X)$ expresses that X is a team on $\mathfrak{A}$ assigning all the free variables of φ.
- $g(\mathfrak{A}, X, V, E, V_0)$ expresses that the game consisting of the positions V and possible moves E where the proponent plays at the positions in $V_0 \subset V$ is the model-checking game $MC(\mathfrak{A}, X, \varphi)$.
- $w(V, E, V_0, S)$ expresses that $S \subset E$ is a positional winning strategy for the opponent.

v, t, g can obviously be chosen as Δ_0-formulas. We choose w as a Σ_1-formula: We have to express that every match which is compatible with the strategy E is finite using a formula $\exists\alpha\exists f \alpha \in \mathrm{On} \wedge f \in \alpha^V \wedge \chi(f)$ where $\chi(f)$ expresses that $f(x) > f(y)$ whenever there is a match compatible with the strategy where one of the players moves from x to y. Notice that ordinals can be defined by a Δ_0-formula expressing that the set is transitive and totally ordered by $\in$ (here we use the axiom of foundation). □

Using the Löweneim-Skolem theorem we have established for inclusion logic validity of a sentence is equivalent to validity on countable structures, or equivalently structures with the natural numbers as a universe. As we will see, this will make the decision problem accessible to the analytical hierarchy. Since in second-order arithmetics there is no Δ^1_1-definition of well-orders (since the set of all encodings of recursive presentations of well-orders is Π^1_1-complete, [18] p. 386), but only definable in Π^1_1, we might expect that we have to go one level up. This is indeed the case: The decision problem of inclusion logic is Π^1_2-complete (in the analytical hierarchy).

Theorem 44. *The decision problem of inclusion logic is in* Π^1_2.

Proof. We can simply repeat the proof of theorem 43 using second-order variables, but we have to replace the Δ_0-statement $\alpha \in \mathrm{On}$ by a Π^1_1-definition of a well-order, which is easily possible by formalising that there exists no infinite descending chain. □

To prove completeness we will use the theory of ω-Turing machines (see for example [3] for details). We will use the following definition:

Definition 12. A (non-deterministic) ω-Turing machine consists of the data of a non-deterministic Turing machine with synchronous, one-sided input and working tapes: $M = (Q, \Sigma, \Gamma, \delta, q_0)$ where Q is the finite set of states, Σ the finite input alphabet, Γ the finite tape alphabet, δ the transition relation and $q_0 \in Q$ the initial state. A word $\alpha \in \Sigma^\omega$ gets accepted by M if and only if there exists a complete non-oscillating run of M on α, that means that every position on the tape gets visited only finitely many times.

We cite the following theorem by Castro and Cucker ([2], proposition 3.5):

Theorem 45. *The set of all (numeric encodings) of ω-Turing machines with input alphabet $\{0,1\}$ who accept every word in $\{0,1\}^\omega$ is Π^1_2-complete.*

Now we will reduce this problem to the decision problem of inclusion logic.

Theorem 46. *The decision problem of inclusion logic is Π^1_2-complete.*

Proof. As we have noticed before there exists a sentence φ of inclusion logic which is only satisfied by structures which are not isomorphic to $(\mathbb{N}, +, \cdot)$. For a Gödel number k of an ω-Turing machine we effectively construct a formula of the form $\psi = \varphi \vee \exists x(\exists 0(0 + 0 = 0 \wedge 0 \subset x) \wedge \exists y(\chi(x, y) \wedge y \subset x))$ over a the signature $\{+, \cdot, A\}$ where A is a unary relation symbol. Of course we can encode any finite initial sequence of a run as an integer and we can choose the encoding such that the empty sequence is encoded by 0. We construct $\chi(x, y)$ as follows:

- x, y encode finite initial sequences of a run of the ω-Turing machine encoded by k on the input "given by A" (the formula An is used to express that the n-th input character is a 1).
- x encodes a proper initial segment of y.
- The tape positions in that part of the sequence encoded by y which does not belong to the sequence encoded by x are all greater then the last tape position in the sequence encoded by x.

Intuitively: At each time when the model-checking game reaches the position "$\exists y$" the proponent has to continue the run of the ω-Turing machine in a manner guaranteeing that it is a complete non-oscillating run. □

Cf. the result by Kleene that every Π^1_1-set (in the natural numbers) can be defined as a least fixed-point of a first-order formula ([4], p. 74, [17], p. 132, [11], p. 48). Thus every Σ^1_1-set can be defined as a greatest fixed-point of a first-order formula, providing an alternative proof of the Π^1_2-completeness (observing effectivity of the translation by Kleene).

10 Generalised semantics and completeness

Just like in general higher-order logic we have a chance to construct a complete proof calculus if we extend our notion of completeness using more general models. In higher-order logic it is possible to get some kinde of completeness by considering models which restrict the interpretation of bound higher-order variables. In logics with team semantics it is natural to consider models restricting the teams which are involved in checking the validity of a formula with respect to a model. Galliani has defined such semantics to prove the completeness of a certain proof calculus.[7] We will use the same notion of generalised semantics but we will apply it to a more general proof calculus.

Definition 13. Given any first-order structure $\mathfrak{A}$ and a set $\mathcal{M}$ of teams over $\mathcal{A}$. A formula φ in a logic with team semantics is *satisfied* by a team $X \in \mathcal{M}$ over the *generalised model* $(\mathfrak{A}, \mathcal{M})$, we will write

$$(\mathfrak{A}, \mathcal{M}) \vDash_X \varphi,$$

if and only if there exists a witnessing family of teams $(X_\psi)_\psi$ for $\mathfrak{A}, X, \varphi$ such that $X_\psi \in \mathcal{M}$ for every occurence of a subformula ψ of φ.

Until now we have guaranteed neither non-triviality nor closedness under any operations of the systems of teams of a generalised model. Since team semantics of first-order formulas with respect to a fixed first-order structured can be described using the unique maximal team satisfying the formula, first-order formulas provide a way to define teams. This motivates the following definition:

Definition 14. A generalised model $(\mathfrak{A}, \mathcal{M})$, where τ is the signature of $\mathfrak{A}$, is said to be *first-order closed* if for every first-order formula φ over the signature $\tau \cup \mathcal{M}$ the maximal team $X \subset A^{fr(\varphi)}$ such that $\mathfrak{A} \models_X \varphi$ is an element of $\mathcal{M}$.

Notice that first-order closedness also implies simple properties like closedness under renaming of variables.

We get the usual *standard semantics* by including every team in $\mathcal{M}$.

Now we will introduce a certain notion of sequents using operations on teams instead of formulas in logics with team semantics. This will make our completeness results more transparent. For simplicity we will only consider the case of I/E-logic.

Definition 15. A *team sequent* is an expression $U \Rightarrow V$ where E, F are finite sets of expressions in a language described as follows: We consider a relational first-order signature τ, a fixed countable set of *team variables* $\mathcal{U}$ and a disjoint countable set of *first-order variables* $\mathcal{V}$. We choose a function $dom : \mathcal{U} \to \mathcal{P}(\mathcal{V})$ such that for every $X \in \mathcal{U}$ the set $dom(X)$ is finite and the fiber of every finite subset of $\mathcal{V}$ is infinite. Now we can inductively define terms and an extension of the function dom to terms:

- The special symbol $\emptyset$ is a term and $dom(\emptyset) := \emptyset$.
- Team variables $X, Y, Z, \ldots \in \mathcal{U}$ are terms.
- For every term t and every $x \in \mathcal{V}$ $t(A/x)$ where A is a fixed special symbol is a term. $dom(t(A/x)) := dom(t) \cup \{x\}$.
- For every term t and every map π of $dom(t)$ into itself $\pi(t)$ is a term. $dom(\pi(t)) = dom(t)$.
- For two terms t, u such that $dom(t) = dom(u)$ the expressions $t \cup u$ and $t \cap u$ are terms and $dom(t \cup u) := dom(t) = dom(u) =: dom(t \cap u)$.
- For every term t and every subset $d \subset dom(t)$ the expression $t \upharpoonright d$ is a term. $dom(t \upharpoonright d) := d$.
- If R is a relation symbol in τ or $=$ with arity k and $\bar{x} \in \mathcal{V}^k$, then $R_{\bar{x}}$ is a term. $dom(R_{\bar{x}})$ is the set of all entries of $\bar{x}$, analogously there are terms $\bar{R}_{\bar{x}}$.

The expressions which can be members of E and F are now given by $t = u$ where t, u are terms such that $dom(t) = dom(u)$. $t \subset u$ is a shorthand for $t \cap u = t$ and $t|u$ is a shorthand for $(t \cap u) \upharpoonright \emptyset = \emptyset$.

Now we can define appropriate semantics:

Definition 16. Let $E \Rightarrow F$ be a team sequent. $\mathcal{U}_E$ and $\mathcal{U}_F$ are the sets of team variables used in E and F respectively. $E \Rightarrow F$ is called *valid with respect to a class K of first-order closed generalised models* if for every $(\mathfrak{A}, \mathcal{M}) \in K$ every interpretation $\mathcal{F} : \mathcal{U}_E \to \mathcal{M}$ such that every element of E is satisfied by $\mathfrak{A}, F$ can be extended to an interpretation $\mathcal{F}' : \mathcal{U}_E \cup \mathcal{U}_F \to \mathcal{M}$ such that every element of F is satisfied by $\mathfrak{A}, \mathcal{F}'$. Satisfaction of an expression by an interpretation $\mathcal{F}$ of its team variables with respect to a structure $\mathfrak{A}$ is defined in the obvious way:

- Every term gets interpreted as an element of $\mathcal{M}$.
- The interpretation of team variables is already given by $\mathcal{F}$.
- $\emptyset$ gets interpreted as the empty set.
- $R_{\bar{x}}$ gets interpreted as $\left\{f \in A^e \mid (f(x_0), \ldots, f(x_{n-1})) \in R^{\mathfrak{A}}\right\}$ where n is the arity of R and e is the set of entries of $\bar{x}$, $\bar{R}_{\bar{x}}$ gets interpreted as the complement of the interpretation of $R_{\bar{x}}$.
- $t \restriction d$ gets interpreted as $X \restriction d$ where X is the interpretation of t.
- $t(A/x)$ gets interpreted as $X(A/x)$ where X is the interpretation of t.
- $\pi(t)$ gets interpreted as the team constructed from the interpretation X of t by mapping the variables using π (formally $X \circ \pi$).
- $\cup, \cap$ get interpreted in the obvious ways as unions and intersections.
- $t = u$ gets interpreted as equality of the interpretations of t, u.

Now we can prove a completeness theorem with respect to those semantics.

Theorem 47. *The set of sequents which are valid with respect to the class of first-order closed generalised models over a fixed countable signature τ is recursively enumerable.*

Proof. Consider a team sequent $E \Rightarrow F$ with sets of variables $\mathcal{U}_E, \mathcal{U}_F$ as before. We will first prove that if $E \Rightarrow F$ is valid with respect to all first-order closed generalised models, then there exists a finite set S consisting of families $(\varphi_X)_{X \in \mathcal{U}_F \setminus \mathcal{U}_E}$ of first-order formulas over the signature $\tau \cup \mathcal{U}_E$ (every team variable $Y \in \mathcal{U}_E$ is considered as a $dom(Y)$-ary relation symbol) where $fr(\varphi_X) = dom(X)$ witnessing the validity of $E \Rightarrow F$ in the following sense: For every first-order closed generalised model $(\mathfrak{A}, \mathcal{M})$ and every interpretation $\mathcal{F} : \mathcal{U}_E \to \mathcal{M}$ satisfying E there is some $(\varphi.) \in S$ such that the extension $\mathcal{F}' : \mathcal{U}_E \cup \mathcal{U}_F \to \mathcal{M}$ defined by $\mathcal{F}'(X) = \{\bar{a} \in A^{dom(X)} \mid \mathcal{A}, \mathcal{U}_E \models \varphi_X(\bar{a})\}$ satisfies F. Assume the contrary: Since F is finite, for every such family $(\varphi.)$ the property of a generalised model $(\mathfrak{A}, \mathcal{M})$ and an interpretation $\mathcal{F} \colon U \to \mathcal{M}$ that the corresponding extension $\mathcal{F}'_{(\varphi.)}$ does not satisfy F can be stated as a first-order sentence $\psi_{(\varphi.)}$ over the signature $\tau \cup \mathcal{U}_E$. There is also a first-order formula θ_E over the same signature expressing the satisfaction of E. Now consider the set $\{\theta_E\} \cup \{\psi_{(\varphi.)} \mid (\varphi.) \text{ is a family as specified before}\}$. By our assumption every finite subset of this set is satisfiable, thus it is satisfiable itself by a first-order structure. Obviously the minimal first-order closed generalised model over this structure is a counterexample for the validity of $E \Rightarrow F$, a contradiction.

Now we can give an algorithm accepting exactly the valid sequents: Given any sequent, search for any proof proving any sentence of the form $\theta_E \to (\neg(\psi_{(\varphi^1_\cdot)} \wedge \ldots \wedge \psi_{(\varphi^n_\cdot)}))$ for some finite set of families $(\varphi^1_\cdot), \ldots, (\varphi^n_\cdot)$ as before. We get a reduction to the completeness theorem for first-order logic. $\square$

This procedure can be used for a procedure for proving sequents in I/E-logic and we get completeness and soundness of this procedure with respect to generalised semantics: To get a team sequent we simply write down all the conditions for witnessing families of teams for the involved formulas.

Example 5. *The formula* $\exists x =\!(x)$ *is equivalent to the formula* $\exists x(\forall y y = x \vee y|x)$ *(cf. [6], theorem 4.19). In our calculus it can be translated to the team sequent* $X = X \Rightarrow X = Y \upharpoonright \emptyset, Z = Y(A/y), Z = I \cup J, I \subset=_{xy}, J \upharpoonright \{x\}|\{y \mapsto x\}(J \upharpoonright \{y\})$ *(where* $dom(Y) = \{x\}$*).*

Theorem 48. *This calculus allows to derive all first-order consequences of I/E-logic.*

Proof. We just have to prove that for every team X in a first-order closed generalised model $\mathcal{M}$ satisfying a given first-order formula φ (in negation normal form) there is a witnessing family of teams in $\mathcal{M}$. This is simply the maximal witnessing family of teams $(M_\cdot)$ (indexed by occurences of subformulas of φ), which can be constructed inductively starting with $M_\varphi = X$ as follows (where $\psi^{\mathfrak{A}}$ denotes the maximal team satisfying ψ):

- If $\psi = \theta \vee \rho$, then $M_\theta := M_\psi \cap \theta^{\mathfrak{A}}$, $M_\rho := M_\psi \cap \rho^{\mathfrak{A}}$.
- If $\psi = \theta \wedge \rho$, then $M_\theta := M_\psi =: M_\rho$.
- If $\psi = \exists x\theta$, then $M_\theta := M_\psi(A/x) \cap \theta^{\mathfrak{A}}$.
- If $\psi = \forall x\theta$, then $M_\theta := M_\psi$.

This family is obviously first-order definable using X.

$\square$

We see how Galliani's approach naturally leads to axiomatisations of first-order consequences, which have been explicitly axiomatised in [14].

10.1 Comparison to the approach by Galliani

We will now have a closer look at the relation of the presented analysis to the concept of generalised semantics by Galliani. His generalised models are also—in our terminology—first-order closed. However, he imposed a second condition of non-triviality, which turned out to be unnecessary to guarantee the derivability of all first-order consequences (in standard semantics): In our terminology he required all singletons $\{\bar{a}\} \subset A^{\bar{x}}$ to be elements of the family of teams $\mathcal{M}$. This is not guaranteed by first-order closedness, for example there are non-trivial structures whose automorphism groups act transitively on the structure. Then there are no singletons which are definable in first-order logic, implying that in a first-order closed generalised model there might be no non-empty teams satisfying $=(x)$. This condition turned out not to be necessary to capture all first-order consequences with respect to standard semantics. This can also be seen from lemma 5.3.8 in [7], which we will now restate in our language (and assuming completeness results):

Lemma 49. *Given any team X in the minimal first-order closed generalised model on $\mathfrak{A}$ containing all singletons such that $\mathfrak{A} \vDash_X \varphi$ where φ is a formula of I/E-logic. X can be defined by a first-order formula γ, $fr(\gamma) = dom(X) \cup d$, and a tuple $\bar{a} \in A^d$ in the following sense: $X = \gamma(\bar{a}) := \{\bar{b} \in A^{dom(X)} \mid \mathfrak{A}, \bar{a}, \bar{b} \vDash \gamma\}$. Then there exists a first-order formula δ with $fr(\delta) = d$ such that for all $\bar{c} \in A^d$ satisfying δ $\mathfrak{A} \vDash_{\gamma(\bar{c})} \varphi$.*

If the teams satisfying φ are closed under unions, then the union of the teams $\gamma(\bar{c})$ is the first-order definable team $X' := \{\bar{b} \in A^{dom(X)} | \mathcal{A} \vDash \exists \bar{x} \delta(\bar{x}) \wedge \gamma(\bar{x}, \bar{b}\}$.

In general, for every provable sequent corresponding to a sequent in the sense of Galliani the corresponding sequent in the sense of Galliani is provable in his calculus. The converse does not hold (for simplicity we assume that we only consider nonempty structures): The tautology $\exists x =(x)$ is true with respect to generalised semantics in the sense of Galliani. However, it is not a tautology with respect to our semantics. It might be interesting to consider how to extend our calculus such that the class of generalised models for which it is complete gets restricted to those also guaranteed to contain all singletons or generalised models providing certain teams defined using Σ^1_1-formulas.

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